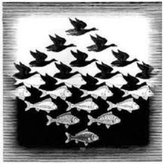
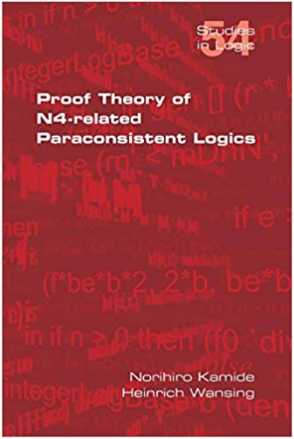
Module 1

Instantiations of (Monotonic) Paraconsistent Reasoning

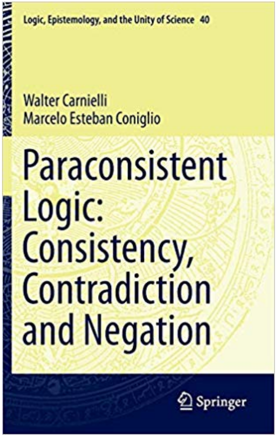


M.C. Escher: *Sky and Water I* 1938 woodcut

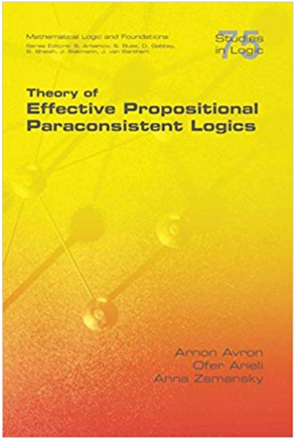
Some Textbooks on Paraconsistent Logics



(a) Kamide, Wansing
Studies in Logic Vol. 54
College Publications, 2015



(b) Carnielli, Coniglio
Logic & Epistemology 40
Springer, 2016



(c) Avron, Arieli, Zamansky
Studies in Logic Vol. 75
College Publications, 2018

Plan of Module 1

1. **Preliminaries; Multi-Valued Logics**
2. Negation and Paraconsistency
3. Maximal Properties of Paraconsistent Logics
    - Maximal Paraconsistency
    - Maximality Relative to CL
4. The Simplest Paraconsistent Multi-Valued Logics (3-Valued Paraconsistent Logics)
5. Combining Paraconsistency and Paracompleteness (4-Valued Paradefinite Logics)
6. A General Construction
7. Other Approaches to Paraconsistency

Some Notations

Propositional language:	$\mathcal{L}$
Atomic formulas (of $\mathcal{L}$):	p, q, r (primed or indexed)
Compound formulas (of $\mathcal{L}$):	ψ, φ, ϕ (primed or indexed)
Sets of formulas (of $\mathcal{L}$):	$\mathcal{S}, \mathcal{T}$ (primed or indexed)
Finite sets of formula (of $\mathcal{L}$):	Γ, Δ (primed or indexed)
The set of atoms in $\mathcal{S}$:	$\text{Atoms}(\mathcal{S})$

What is a (propositional) Logic?

A (Tarskian) *consequence relation* $\vdash$ for a language $\mathcal{L}$:

- Reflexivity: $\psi \vdash \psi$.
- Monotonicity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{S}' \vdash \psi$.
- Transitivity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S}', \psi \vdash \varphi$ then $\mathcal{S}, \mathcal{S}' \vdash \varphi$.

A consequence relation $\vdash$ is called:

- Structural: if $\mathcal{S} \vdash \psi$ then $\theta(\mathcal{S}) \vdash \theta(\psi)$ for every $\mathcal{L}$-substitution θ.
- Non-trivial: $\mathcal{S} \not\vdash \psi$ for some $\mathcal{S} \neq \emptyset$.
- Finitary: if $\mathcal{S} \vdash \psi$ then $\Gamma \vdash \psi$ for some finite $\Gamma \subseteq \mathcal{S}$.

A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where

- $\mathcal{L}$ is a propositional language, and
- $\vdash$ is a structural, non-trivial and finitary consequence relation for $\mathcal{L}$.

What is a (propositional) Logic?

A (Tarskian) *consequence relation* $\vdash$ for a language $\mathcal{L}$:

- Reflexivity: $\psi \vdash \psi$.
- Monotonicity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{S}' \vdash \psi$.
- Transitivity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S}', \psi \vdash \varphi$ then $\mathcal{S}, \mathcal{S}' \vdash \varphi$.

A consequence relation $\vdash$ is called:

- Structural: if $\mathcal{S} \vdash \psi$ then $\theta(\mathcal{S}) \vdash \theta(\psi)$ for every $\mathcal{L}$-substitution θ.
- Non-trivial: $\mathcal{S} \not\vdash \psi$ for some $\mathcal{S} \neq \emptyset$.
- Finitary: if $\mathcal{S} \vdash \psi$ then $\Gamma \vdash \psi$ for some finite $\Gamma \subseteq \mathcal{S}$.

A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where

- $\mathcal{L}$ is a propositional language, and
- $\vdash$ is a structural, non-trivial and finitary consequence relation for $\mathcal{L}$.

Note: In this module we assume *monotonic reasoning*.

Basic Connectives of Propositional Languages

Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic.

$\wedge$ is a *conjunction* for $\mathfrak{L}$ if $\mathcal{S} \vdash \psi \wedge \varphi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \varphi$.

$\vee$ is a *disjunction* for $\mathfrak{L}$ if $\mathcal{S}, \psi \vee \varphi \vdash \sigma$ iff $\mathcal{S}, \psi \vdash \sigma$ and $\mathcal{S}, \varphi \vdash \sigma$.

$\supset$ is an *implication* for $\mathfrak{L}$ if $\mathcal{S}, \varphi \vdash \psi$ iff $\mathcal{S} \vdash \varphi \supset \psi$.

Basic Connectives of Propositional Languages

Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic.

$\wedge$ is a *conjunction* for $\mathfrak{L}$ if $\mathcal{S} \vdash \psi \wedge \varphi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \varphi$.
(Equivalently, $\mathcal{S}, \psi \wedge \phi \vdash \tau \;\Leftrightarrow \mathcal{S}, \psi, \phi \vdash \tau$)

$\vee$ is a *disjunction* for $\mathfrak{L}$ if $\mathcal{S}, \psi \vee \varphi \vdash \sigma$ iff $\mathcal{S}, \psi \vdash \sigma$ and $\mathcal{S}, \varphi \vdash \sigma$.
(Equivalently, if $\vdash$ is multi-conclusioned, $\mathcal{S} \vdash \psi \vee \phi \;\Leftrightarrow \mathcal{S} \vdash \psi, \phi$)

$\supset$ is an *implication* for $\mathfrak{L}$ if $\mathcal{S}, \varphi \vdash \psi$ iff $\mathcal{S} \vdash \varphi \supset \psi$.
(Inferences to theoremhood: $\psi_1, \ldots, \psi_n \vdash \phi \;\Leftrightarrow \vdash \psi_1 \supset (\psi_2 \ldots \supset (\psi_n \supset \phi))$)

Basic Connectives of Propositional Languages

Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic.

$\wedge$ is a *conjunction* for $\mathfrak{L}$ if $\mathcal{S} \vdash \psi \wedge \varphi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \varphi$.
(Equivalently, $\mathcal{S}, \psi \wedge \phi \vdash \tau \;\Leftrightarrow \mathcal{S}, \psi, \phi \vdash \tau$)

$\vee$ is a *disjunction* for $\mathfrak{L}$ if $\mathcal{S}, \psi \vee \varphi \vdash \sigma$ iff $\mathcal{S}, \psi \vdash \sigma$ and $\mathcal{S}, \varphi \vdash \sigma$.
(Equivalently, if $\vdash$ is multi-conclusioned, $\mathcal{S} \vdash \psi \vee \phi \;\Leftrightarrow \mathcal{S} \vdash \psi, \phi$)

$\supset$ is an *implication* for $\mathfrak{L}$ if $\mathcal{S}, \varphi \vdash \psi$ iff $\mathcal{S} \vdash \varphi \supset \psi$.
(Inferences to theoremhood: $\psi_1, \ldots, \psi_n \vdash \phi \;\Leftrightarrow \vdash \psi_1 \supset (\psi_2 \ldots \supset (\psi_n \supset \phi))$)

$\mathfrak{L}$ is *semi-normal* if it has (at least) one of these connectives.

$\mathfrak{L}$ is *normal* if it has *all* the three connectives.

Multi-Valued Matrices and Their Logics

The most standard way of defining logics is by matrices.

A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

- $\mathcal{V}$: the set of truth values,
- $\mathcal{D} \subset \mathcal{V}$: the *designated* elements of $\mathcal{V}$,
- $\mathcal{O}$: the interpretations (the 'truth tables') of the connectives (an *n*-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ for every *n*-ary connective $\diamond$ of $\mathcal{L}$).

Multi-Valued Matrices and Their Logics

The most standard way of defining logics is by matrices.

A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

- $\mathcal{V}$: the set of truth values,
- $\mathcal{D} \subset \mathcal{V}$: the *designated* elements of $\mathcal{V}$,
- $\mathcal{O}$: the interpretations (the 'truth tables') of the connectives (an *n*-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ for every *n*-ary connective $\diamond$ of $\mathcal{L}$).

The induced semantics:

- An *$\mathcal{M}$-valuation* a function $\nu : \mathsf{WFF}(\mathcal{L}) \to \mathcal{V}$ such that $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n))$ for every connective $\diamond$.
- The *$\mathcal{M}$-models of a formulas ψ*: $\mathsf{mod}_{\mathcal{M}}(\psi) = \{\nu \mid \nu(\psi) \in \mathcal{D}\}$.
- The *$\mathcal{M}$-models of a set* Γ: $\mathsf{mod}_{\mathcal{M}}(\mathcal{S}) = \bigcap_{\psi \in \mathcal{S}} \mathsf{mod}_{\mathcal{M}}(\psi)$.

Multi-Valued Matrices and Their Logics

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ – a matrix for $\mathcal{L}$. The induced logic $\mathfrak{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$:

$\mathcal{S} \vdash_{\mathcal{M}} \psi$ iff $\mathrm{mod}_{\mathcal{M}}(\mathcal{S}) \subseteq \mathrm{mod}_{\mathcal{M}}(\psi)$.

Multi-Valued Matrices and Their Logics

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ – a matrix for $\mathcal{L}$. The induced logic $\mathfrak{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$:

$\mathcal{S} \vdash_{\mathcal{M}} \psi$ iff $\mathsf{mod}_{\mathcal{M}}(\mathcal{S}) \subseteq \mathsf{mod}_{\mathcal{M}}(\psi)$.

Proposition (Shoesmith & Smiley, 1971)

For every propositional language $\mathcal{L}$ and every finite matrix $\mathcal{M}$ (for $\mathcal{L}$), $\mathfrak{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.

D.J.Shoesmith and T.J. Smiley. *Deducibility and many-valuedness*. Journal of Symbolic Logic 36, pp.610–622, 1971

Semantic Definitions of the Basic Connectives

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ – a matrix for $\mathcal{L}$.

$\wedge$ in $\mathcal{L}$ is an *$\mathcal{M}$-conjunction*: $\forall a, b \in \mathcal{V}$, $a \mathbin{\widetilde{\wedge}} b \in \mathcal{D}$ iff $a \in \mathcal{D}$ and $b \in \mathcal{D}$.

$\vee$ in $\mathcal{L}$ is an *$\mathcal{M}$-disjunction*: $\forall a, b \in \mathcal{V}$, $a \mathbin{\widetilde{\vee}} b \in \mathcal{D}$ iff $a \in \mathcal{D}$ or $b \in \mathcal{D}$.

$\supset$ in $\mathcal{L}$ is an *$\mathcal{M}$-implication*: $\forall a, b \in \mathcal{V}$, $a \mathbin{\widetilde{\supset}} b \in \mathcal{D}$ iff $a \notin \mathcal{D}$ or $b \in \mathcal{D}$.

Semantic Definitions of the Basic Connectives

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ – a matrix for $\mathcal{L}$.

$\wedge$ in $\mathcal{L}$ is an *$\mathcal{M}$-conjunction*: $\forall a, b \in \mathcal{V}$, $a \,\tilde{\wedge}\, b \in \mathcal{D}$ iff $a \in \mathcal{D}$ and $b \in \mathcal{D}$.

$\vee$ in $\mathcal{L}$ is an *$\mathcal{M}$-disjunction*: $\forall a, b \in \mathcal{V}$, $a \,\tilde{\vee}\, b \in \mathcal{D}$ iff $a \in \mathcal{D}$ or $b \in \mathcal{D}$.

$\supset$ in $\mathcal{L}$ is an *$\mathcal{M}$-implication*: $\forall a, b \in \mathcal{V}$, $a \,\tilde{\supset}\, b \in \mathcal{D}$ iff $a \notin \mathcal{D}$ or $b \in \mathcal{D}$.

An $\mathcal{M}$-conjunction [$\mathcal{M}$-disjunction, $\mathcal{M}$-implication] is also a conjunction [disjunction, implication] for $\mathfrak{L}_{\mathcal{M}}$.

If $\mathcal{M}$ has an $\mathcal{M}$-conjunction, or an $\mathcal{M}$-disjunction, or an $\mathcal{M}$-implication, then $\mathfrak{L}_{\mathcal{M}}$ is semi-normal. If $\mathcal{M}$ has all of them then $\mathfrak{L}_{\mathcal{M}}$ is normal.

Classical Logic

$$\text{CL} = \langle \{t, f\}, \{t\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f
t	t	t
f	t	f

$\tilde{\wedge}$	t	f
t	t	f
f	f	f

$\tilde{\neg}$	
t	f
f	t

Some Properties:

- Not (pre-)paracomplete:
  [LEM]: if $\mathcal{S}, \psi \vdash_{\text{CL}} \varphi$ and $\mathcal{S}, \neg\psi \vdash_{\text{CL}} \varphi$ then $\mathcal{S} \vdash_{\text{CL}} \varphi$
  (Alternatively, $\vdash_{\text{CL}} \psi \vee \neg\psi$]).
- Not (pre-)paraconsistent: $\psi, \neg\psi \vdash_{\text{CL}} \varphi$.

Kleene's Logic

$$\mathsf{KL} = \langle \{t, f, \bot\}, \{t\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\bot$
t	t	t	t
f	t	f	$\bot$
$\bot$	t	$\bot$	$\bot$

$\tilde{\wedge}$	t	f	$\bot$
t	t	f	$\bot$
f	f	f	f
$\bot$	$\bot$	f	$\bot$

$\tilde{\neg}$	
t	f
f	t
$\bot$	$\bot$

Some Properties:

- (pre-)paracomplete:
  $\mathcal{S}, \psi \vdash_{\mathsf{KL}} \varphi$ and $\mathcal{S}, \neg\psi \vdash_{\mathsf{KL}} \varphi$ does *not* imply that $\mathcal{S} \vdash_{\mathsf{KL}} \varphi$
  [$\nvdash_{\mathsf{KL}} \psi \vee \neg\psi$]
- Not (pre-)paraconsistent: $\psi, \neg\psi \vdash_{\mathsf{KL}} \varphi$.

S. C. Kleene. *Introduction to Metamathematics*. Van Nostrand, 1950.

Asenjo-Priest's Logic

$$\mathsf{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

Some Properties:

- Not (pre-)paracomplete:
  if $\mathcal{S}, \psi \vdash_{\mathsf{LP}} \varphi$ and $\mathcal{S}, \neg\psi \vdash_{\mathsf{LP}} \varphi$ then $\mathcal{S} \vdash_{\mathsf{LP}} \varphi$ [$\vdash_{\mathsf{LP}} \psi \vee \neg\psi$].
- (pre-)paraconsistent: $\psi, \neg\psi \nvdash_{\mathsf{LP}} \varphi$.

F. G. Asenjo. *A calculus of antinomies*. Notre Dame Journal of Formal Logic, 7:103–106, 1966.

G. Priest. *Logic of paradox*. Journal of Philosophical Logic, 8:219–241, 1979.

G. Priest. *Reasoning about truth*. Artificial Intelligence, 39:231–244, 1989.

Dunn-Belnap's Logic

$$\mathsf{FDE} = \langle \{t, f, \top, \bot\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\tilde{\wedge}$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

Some Properties:

- (pre-)paracomplete:
  $\mathcal{S}, \psi \vdash_{\mathsf{FDE}} \varphi$ and $\mathcal{S}, \neg\psi \vdash_{\mathsf{FDE}} \varphi$ does *not* imply that $\mathcal{S} \vdash_{\mathsf{FDE}} \varphi$
  [$\nvdash_{\mathsf{FDE}} \psi \vee \neg\psi$]
- (pre-)paraconsistent: $\psi, \neg\psi \nvdash_{\mathsf{FDE}} \varphi$.

J. M. Dunn. *Intuitive semantics for first-degree entailments and 'coupled trees'*. Philosophical Studies, 29:149–168, 1976.

N. D. Belnap. *How a computer should think*. Contemporary Aspects of Philosophy, pages 30–56, Oriel Press, 1977.

N. D. Belnap. *A useful four-valued logic* . Modern Uses of Multiple-Valued Logics, pages 7–37, Reidel, 1977.

Plan of Module 1

1. Preliminaries; Multi-Valued Logics
2. **Negation and Paraconsistency**
3. Maximal Properties of Paraconsistent Logics
   - Maximal Paraconsistency
   - Maximality Relative to CL
4. The Simplest Paraconsistent Multi-Valued Logics (3-Valued Paraconsistent Logics)
5. Combining Paraconsistency and Paracompleteness (4-Valued Paradefinite Logics)
6. A General Construction
7. Other Approaches to Paraconsistency

Inconsistency Handling

- *Coherent Approaches*
  - Consistency restoration in the presence of contradictions
  - Information is revised (belief revision, DB repair, . . .)
- *Paraconsistent Approaches*
  - Reasoning with inconsistent premises
  - No information loss
  - Inference should not be trivialized (no data explosion)

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

The principle of explosion ('ex contradictione sequitur quodlibet'):

> *If one claims something is both true and not true, one can logically derive any conclusion.*

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

The principle of explosion ('ex contradictione sequitur quodlibet'):

> *If one claims something is both true and not true, one can logically derive any conclusion.*

$$\psi, \neg\psi \vdash \phi$$

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

The principle of explosion ('ex contradictione sequitur quodlibet'):

If one claims something is both true and not true, one can logically derive any conclusion.

$$\psi, \neg\psi \vdash \phi$$

However, by this principle,

- Any inconsistent theory becomes trivial, and so
- No sensible reasoning can take place in the presence of contradictions.

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

The principle of explosion ('ex contradictione sequitur quodlibet'):

If one claims something is both true and not true, one can logically derive any conclusion.

$$\psi, \neg\psi \vdash \phi$$

However, by this principle,

- Any inconsistent theory becomes trivial, and so
- No sensible reasoning can take place in the presence of contradictions.

Rejecting this principle means the rejection at least one of:

- *Disjunction Introduction*: from infer ψ infer $\psi \vee \phi$
- *The Disjunctive Syllogism*: from $\neg\psi$ and $\psi \vee \phi$ infer ϕ

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

Paraconsistent logics do allow non-trivial inconsistent theories:
$\langle \mathcal{L}, \vdash \rangle$ is *pre* $\neg$*-paraconsistent* if there are ψ, ϕ such that $\psi, \neg\psi \not\vdash \phi$.

(By structurality, it is enough that there are *atoms* p, q s.t. $p, \neg p \not\vdash q$)

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

Paraconsistent logics do allow non-trivial inconsistent theories:
$\langle \mathcal{L}, \vdash \rangle$ is *pre* $\neg$*-paraconsistent* if there are ψ, ϕ such that $\psi, \neg\psi \nvdash \phi$.

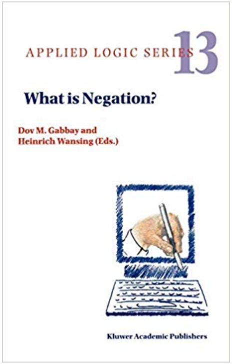
(By structurality, it is enough that there are *atoms* p, q s.t. $p, \neg p \nvdash q$)

Paraconsistency is characterized by a 'negation connective'. But there is no general agreement about the properties of such a connective.

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

Paraconsistent logics do allow non-trivial inconsistent theories:
$\langle \mathcal{L}, \vdash \rangle$ is *pre* $\neg$*-paraconsistent* if there are ψ, ϕ such that $\psi, \neg\psi \nvdash \phi$.
(By structurality, it is enough that there are *atoms* p, q s.t. $p, \neg p \nvdash q$)

Paraconsistency is characterized by a 'negation connective'. But there is no general agreement about the properties of such a connective.



What is Negation?

Intuition (a minimal condition): A logic with a negation connective should not admit any entailment that is excluded by classical logic.

What is Negation?

Intuition (a minimal condition): A logic with a negation connective should not admit any entailment that is excluded by classical logic.

- $\neg$-containment in classical logic.
  The logic does not allow negation rules that are excluded by CL.

  $\mathsf{CL} = \langle \{t, f\}, \{t\}, \{\tilde{\neg}, \ldots\} \rangle$

  $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is $\neg$-contained in classical logic if: $\Gamma \vdash \psi \ \Rightarrow \ \Gamma \vdash_{\mathsf{CL}} \psi$

What is Negation?

Intuition (a minimal condition): A logic with a negation connective should not admit any entailment that is excluded by classical logic.

- ¬-containment in classical logic.
  The logic does not allow negation rules that are excluded by CL.

  $\mathsf{CL} = \langle \{t, f\}, \{t\}, \{\tilde{\neg}, \ldots\} \rangle$

  $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is ¬-contained in classical logic if: $\Gamma \vdash \psi \;\Rightarrow\; \Gamma \vdash_{\mathsf{CL}} \psi$

- ¬-coherence with classical logic.
  The logic has a semi-normal fragment which is ¬-contained in CL.

What is Negation?

Intuition (a minimal condition): A logic with a negation connective should not admit any entailment that is excluded by classical logic.

- ¬-containment in classical logic.
  The logic does not allow negation rules that are excluded by CL.

  $\mathsf{CL} = \langle \{t, f\}, \{t\}, \{\tilde{\neg}, \ldots\} \rangle$

  $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is ¬-contained in classical logic if: $\Gamma \vdash \psi \;\Rightarrow\; \Gamma \vdash_{\mathsf{CL}} \psi$

- ¬-coherence with classical logic.
  The logic has a semi-normal fragment which is ¬-contained in CL.

- ¬ is a negation for $\mathfrak{L}$ ¬-coherent with classical logic.

What is Negation?

Intuition (a minimal condition): A logic with a negation connective should not admit any entailment that is excluded by classical logic.

- $\neg$-containment in classical logic.
  The logic does not allow negation rules that are excluded by CL.

  $\mathsf{CL} = \langle \{t, f\}, \{t\}, \{\tilde{\neg}, \ldots\} \rangle$

  $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is $\neg$-contained in classical logic if: $\Gamma \vdash \psi \;\Rightarrow\; \Gamma \vdash_{\mathsf{CL}} \psi$

- $\neg$-coherence with classical logic.
  The logic has a semi-normal fragment which is $\neg$-contained in CL.

- $\neg$ is a negation for $\mathfrak{L}$ $\neg$-coherent with classical logic.

Note: If $\neg$ is a negation for $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, then for any atom p it holds that $p \nvdash \neg p$ and $\neg p \nvdash p$.

Definition

A logic $\mathfrak{L}$ is ¬-paraconsistent if it is pre-paraconsistent ($p, \neg p \nvdash q$) and $\neg$ is a negation for $\mathfrak{L}$.

Paraconsistemt Logics
(Vesiliev, Łukasiewicz, Jaśkowski, da-Costa, Nelson, Anderson, Belnap, . . .)

Definition

A logic $\mathfrak{L}$ is $\neg$-paraconsistent if it is pre-paraconsistent ($p, \neg p \nvdash q$) and $\neg$ is a negation for $\mathfrak{L}$.

Note: Three inherent conditions in the definition of paraconsistency:

- Pre-paraconsistency (to avoid explosion)
- A proper behavior of the underlying unary connective $\neg$
- Minimal expressive power (semi-normality)

Plan of Module 1

1. Preliminaries; Multi-Valued Logics
2. Negation and Paraconsistency
3. **Maximal Properties of Paraconsistent Logics**
   - **Maximal Paraconsistency**
   - **Maximality Relative to CL**
4. The Simplest Paraconsistent Multi-Valued Logics
   (3-Valued Paraconsistent Logics)
5. Combining Paraconsistency and Paracompleteness
   (4-Valued Paradefinite Logics)
6. A General Construction
7. Other Approaches to Paraconsistency

Maximal Paraconsistency

The two requirements from a paraconsistent logic (pre-paraconsistency and ¬-coherence with CL) are usually not enough.

A useful paraconsistent logic should be maximal (da-Costa, 1974).

N. C. A. da Costa. *On the theory of inconsistent formal systems*. Notre Dame Journal of Formal Logic, 15:497–510, 1974.

Maximal Paraconsistency

The two requirements from a paraconsistent logic (pre-paraconsistency and $\neg$-coherence with CL) are usually not enough.

A useful paraconsistent logic should be maximal (da-Costa, 1974).

Intuition: By trying to further extend a paraconsistent logic (without changing the language), paraconsistency is lost.

N. C. A. da Costa. *On the theory of inconsistent formal systems*. Notre Dame Journal of Formal Logic, 15:497–510, 1974.

Maximal Paraconsistency

The two requirements from a paraconsistent logic (pre-paraconsistency and $\neg$-coherence with CL) are usually not enough.

A useful paraconsistent logic should be maximal (da-Costa, 1974).

<u>Intuition</u>: By trying to further extend a paraconsistent logic (without changing the language), paraconsistency is lost.

A logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is *maximally paraconsistent*, if it is paraconsistent, and every logic $\mathfrak{L}' = \langle \mathcal{L}, \Vdash \rangle$ that properly extends $\mathfrak{L}$ (that is, $\vdash \subsetneq \Vdash$) is *not* paraconsistemt.

N. C. A. da Costa. *On the theory of inconsistent formal systems*. Notre Dame Journal of Formal Logic, 15:497–510, 1974.

Maximality Relative to Classical Logic

Maximal paraconsistenct is an ***absolute*** criterion. Another way of interpreting maximality is ***relative***, with respect to a reference logic. Classical logic is a natural candidate for this.

Maximality Relative to Classical Logic

Maximal paraconsistenct is an ***absolute*** criterion. Another way of interpreting maximality is ***relative***, with respect to a reference logic. Classical logic is a natural candidate for this.

Intuition: A useful paraconsistent logic should retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories.

Maximality Relative to Classical Logic

Maximal paraconsistenct is an ***absolute*** criterion. Another way of interpreting maximality is ***relative***, with respect to a reference logic. Classical logic is a natural candidate for this.

Intuition: A useful paraconsistent logic should retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories.

A logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is *maximal relative to classical logic*, if:

- $\mathfrak{L}$ is $\neg$-contained in classical logic
  ($\Gamma \vdash \psi \;\Rightarrow\; \Gamma \vdash_{\mathsf{CL}} \psi$, where CL $= \langle \{t, f\}, \{t\}, \{\tilde{\neg}, \ldots\} \rangle$)
- Let ψ be a CL-tautology not provable in $\mathfrak{L}$. Then by adding a ψ to $\mathfrak{L}$ as a new axiom schema, *all* the CL-tautologies become provable in $\mathfrak{L}$.

Plan of Module 1

1. Preliminaries; Multi-Valued Logics
2. Negation and Paraconsistency
3. Maximal Properties of Paraconsistent Logics
   - Maximal Paraconsistency
   - Maximality Relative to CL
4. **The Simplest Paraconsistent Multi-Valued Logics (3-Valued Paraconsistent Logics)**
5. Combining Paraconsistency and Paracompleteness (4-Valued Paradefinite Logics)
6. A General Construction
7. Other Approaches to Paraconsistency

Three-Valued Paraconsistent Logics

- One of the oldest and best known approaches to paraconsistency
- The simplest multivalued framework for paraconsistent reasoning.

Three-Valued Paraconsistent Logics

- One of the oldest and best known approaches to paraconsistency
- The simplest multivalued framework for paraconsistent reasoning.

Indeed:

- Every paraconsistent matrix which is $\neg$-contained in classical logic has at least two designated elements.
- No two-valued matrix which is $\neg$-contained in classical logic can be paraconsistent.

Three-Valued Paraconsistent Logics

- One of the oldest and best known approaches to paraconsistency
- The simplest multivalued framework for paraconsistent reasoning.

Indeed:

- Every paraconsistent matrix which is $\neg$-contained in classical logic has at least two designated elements.
- No two-valued matrix which is $\neg$-contained in classical logic can be paraconsistent.

Lemma: Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language with $\neg$.

- If $\mathcal{L}_{\mathcal{M}}$ is $\neg$-contained in CL, there is $t \in \mathcal{V}$ s.t. $t \in \mathcal{D}$ and $\tilde{\neg} t \notin \mathcal{D}$.
- $\mathcal{M}$ is pre-paraconsistent iff there is $\top \in \mathcal{V}$ s.t. $\top \in \mathcal{D}$ and $\tilde{\neg} \top \in \mathcal{D}$.

Three-Valued Paraconsistent Logics

- One of the oldest and best known approaches to paraconsistency
- The simplest multivalued framework for paraconsistent reasoning.

Indeed:

- Every paraconsistent matrix which is $\neg$-contained in classical logic has at least two designated elements.
- No two-valued matrix which is $\neg$-contained in classical logic can be paraconsistent.

Lemma: Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language with $\neg$.

- If $\mathcal{L}_{\mathcal{M}}$ is $\neg$-contained in CL, there is $t \in \mathcal{V}$ s.t. $t \in \mathcal{D}$ and $\tilde{\neg} t \notin \mathcal{D}$.
- $\mathcal{M}$ is pre-paraconsistent iff there is $\top \in \mathcal{V}$ s.t. $\top \in \mathcal{D}$ and $\tilde{\neg} \top \in \mathcal{D}$.

Proof: Since $p \nvdash_{CL} \neg p$, also $p \nvdash_{\mathcal{M}} \neg p$, and so there is some $t \in \mathcal{D}$, such that $\tilde{\neg} t \notin \mathcal{D}$. Since $\mathcal{M}$ is pre-paraconsistent, $p, \neg p \nvdash_{\mathcal{M}} q$, and so there is some $\top \in \mathcal{D}$ such that $\tilde{\neg} \top \in \mathcal{D}$. $\square$

Three-Valued Paraconsistent Logics (Cot'd.)

Proposition

Let $\mathcal{M}$ be a 3-valued matrix such that $\mathfrak{L}_{\mathcal{M}}$ is paraconsistent.
Then $\mathcal{M}$ is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which
$\mathcal{V} = \{t, f, \top\}$, $\mathcal{D} = \{t, \top\}$, *and* $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top \in \mathcal{D}$.

Three-Valued Paraconsistent Logics (Cot'd.)

Proposition

Let $\mathcal{M}$ be a 3-valued matrix such that $\mathfrak{L}_{\mathcal{M}}$ is paraconsistent. Then $\mathcal{M}$ is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, f, \top\}$, $\mathcal{D} = \{t, \top\}$, and $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top \in \mathcal{D}$.

Two possible 3-valued negation connectives:

- Sette's negation: $\tilde{\neg} t = f, \;\; \tilde{\neg} f = t, \;\; \tilde{\neg} \top = t$
- Kleene's negation: $\tilde{\neg} t = f, \;\; \tilde{\neg} f = t, \;\; \tilde{\neg} \top = \top$

Three-Valued Paraconsistent Logics (Cot'd.)

Proposition

Let $\mathcal{M}$ be a 3-valued matrix such that $\mathfrak{L}_{\mathcal{M}}$ is paraconsistent.
Then $\mathcal{M}$ is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which
$\mathcal{V} = \{t, f, \top\}$, $\mathcal{D} = \{t, \top\}$, *and* $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top \in \mathcal{D}$.

Two possible 3-valued negation connectives:

- Sette's negation: $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top = t$
- Kleene's negation: $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top = \top$

Proposition (maximality of 3-valued paraconsistent logics)

Every 3-valued (semi-normal) paraconsistent logic that is $\neg$-contained in CL, is maximal in both senses.*

* O.Arieli, A.Avron. *Three-valued paraconsistent propositional logics*. New Directions in Paraconsistent Logics, pp.91-129, 2015.

** In the 3-valued case maximal consistency of normal paraconsistent logic implies maximality relative to CL.

*** Semi-normality is important here: a 3-valued logic with only Kleene's negation is not maximally paraconsistent.

Sette's Logic

$$\mathsf{P}_1 = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\supset}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

A. M. Sette. *On propositional calculus P1*. Mathematica Japonica, 16:173–180, 1973.

Notable 3-valued Paraconsistent Logics

Sette's Logic

$$\mathsf{P}_1 = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\}\rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\supset}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

Pros:

- Paraconsistent and normal
- Maximal in both senses

A. M. Sette. *On propositional calculus P1*. Mathematica Japonica, 16:173–180, 1973.

Sette's Logic

$$\mathsf{P_1} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\supset}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

Pros:

- Paraconsistent and normal
- Maximal in both senses

Cons:

- $\neg$ is not right involutive: $p \nvdash_{\mathsf{P_1}} \neg\neg p$

A. M. Sette. *On propositional calculus P1*. Mathematica Japonica, 16:173–180, 1973.

Sette's Logic

$$\mathsf{P}_1 = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\supset}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

Pros:

- Paraconsistent and normal
- Maximal in both senses

Cons:

- $\neg$ is not right involutive: $p \nvdash_{\mathsf{P}_1} \neg\neg p$
- Some unintuitive entailments: $\neg p \nvdash_{\mathsf{P}_1} \neg(p \wedge q)$

A. M. Sette. *On propositional calculus P1*. Mathematica Japonica, 16:173–180, 1973.

Sette's Logic

$$\mathsf{P}_1 = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\}\rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\supset}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

Pros:

- Paraconsistent and normal
- Maximal in both senses

Cons:

- $\neg$ is not right involutive: $p \nvdash_{\mathsf{P}_1} \neg\neg p$
- Some unintuitive entailments: $\neg p \nvdash_{\mathsf{P}_1} \neg(p \wedge q)$
- Explosive w.r.t. negative data: $\neg p, \neg\neg p \vdash_{\mathsf{P}_1} q$

A. M. Sette. *On propositional calculus P1*. Mathematica Japonica, 16:173–180, 1973.

Notable 3-valued Paraconsistent Logics

Asenjo-Priest's Logic

$$\text{LP} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\}\rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

F. G. Asenjo. *A calculus of antinomies*. Notre Dame Journal of Formal Logic, 7:103–106, 1966.

G. Priest. *Logic of paradox*. Journal of Philosophical Logic, 8:219–241, 1979.

G. Priest. *Reasoning about truth*. Artificial Intelligence, 39:231–244, 1989.

Asenjo-Priest's Logic

$$\text{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

- Like (strong) Kleene's logic, but with a designated middle element

F. G. Asenjo. *A calculus of antinomies*. Notre Dame Journal of Formal Logic, 7:103–106, 1966.

G. Priest. *Logic of paradox*. Journal of Philosophical Logic, 8:219–241, 1979.

G. Priest. *Reasoning about truth*. Artificial Intelligence, 39:231–244, 1989.

Asenjo-Priest's Logic

$$\text{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

- Like (strong) Kleene's logic, but with a designated middle element
- Paraconsistent

F. G. Asenjo. *A calculus of antinomies*. Notre Dame Journal of Formal Logic, 7:103–106, 1966.

G. Priest. *Logic of paradox*. Journal of Philosophical Logic, 8:219–241, 1979.

G. Priest. *Reasoning about truth*. Artificial Intelligence, 39:231–244, 1989.

Asenjo-Priest's Logic

$$\text{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

- Like (strong) Kleene's logic, but with a designated middle element
- Paraconsistent
- Maximal in both senses

F. G. Asenjo. *A calculus of antinomies*. Notre Dame Journal of Formal Logic, 7:103–106, 1966.

G. Priest. *Logic of paradox*. Journal of Philosophical Logic, 8:219–241, 1979.

G. Priest. *Reasoning about truth*. Artificial Intelligence, 39:231–244, 1989.

Notable 3-valued Paraconsistent Logics

Asenjo-Priest's Logic

$$\text{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

- Like (strong) Kleene's logic, but with a designated middle element
- Paraconsistent
- Maximal in both senses
- Has the same tautologies as those of CL: $\vdash_{\text{LP}} \psi$ iff $\vdash_{\text{CL}} \psi$

F. G. Asenjo. *A calculus of antinomies*. Notre Dame Journal of Formal Logic, 7:103–106, 1966.

G. Priest. *Logic of paradox*. Journal of Philosophical Logic, 8:219–241, 1979.

G. Priest. *Reasoning about truth*. Artificial Intelligence, 39:231–244, 1989.

Notable 3-valued Paraconsistent Logics

Asenjo-Priest's Logic

$$\text{LP} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\}\rangle$$

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

- Like (strong) Kleene's logic, but with a designated middle element
- Paraconsistent
- Maximal in both senses
- Has the same tautologies as those of CL: $\vdash_{\text{LP}} \psi$ iff $\vdash_{\text{CL}} \psi$
- Semi-normal but not normal (an implication is not definable)

F. G. Asenjo. *A calculus of antinomies*. Notre Dame Journal of Formal Logic, 7:103–106, 1966.

G. Priest. *Logic of paradox*. Journal of Philosophical Logic, 8:219–241, 1979.

G. Priest. *Reasoning about truth*. Artificial Intelligence, 39:231–244, 1989.

Extensions of LP by Propositional Constants

- LP^{B} – extension of LP to $\{\neg, \vee, \wedge, \mathsf{B}\}$ (where $\nu(\mathsf{B}) = \top$)
- LP^{F} – extension of LP to $\{\neg, \vee, \wedge, \mathsf{F}\}$ (where $\nu(\mathsf{F}) = f$)
- $\mathsf{LP}^{\mathsf{F,B}}$ – extension of LP to $\{\neg, \vee, \wedge, \mathsf{F}, \mathsf{B}\}$

Extensions of LP by Propositional Constants

- LP^{B} – extension of LP to $\{\neg, \vee, \wedge, \mathsf{B}\}$ (where $\nu(\mathsf{B}) = \top$)
- LP^{F} – extension of LP to $\{\neg, \vee, \wedge, \mathsf{F}\}$ (where $\nu(\mathsf{F}) = f$)
- $\mathsf{LP}^{\mathsf{F,B}}$ – extension of LP to $\{\neg, \vee, \wedge, \mathsf{F}, \mathsf{B}\}$

- LP^{B} and $\mathsf{LP}^{\mathsf{F,B}}$ are $\neg$-coherent with CL
- LP and LP^{F} are $\neg$-contained in CL
- LP and LP^{F} have the same valid formulas as in CL
- all the extensions are still not normal (implications are not definable in them)

PAC (RM_3) – Extension of LP by Implication

(Avron, Batens, da Costa, D'Ottaviano)

$$\text{PAC} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\} \rangle$$

A. Avron. *On an implication connective of RM*. Notre Dame Journal of Formal Logic, 27:201–209, 1986.

D. Batens. *Paraconsistent extensional propositional logics*. Logique et Analyse, 90–91:195–234, 1980.

I. D'Ottaviano. *The completeness and compactness of a three-valued first-order logic*. Revista Colombiana de Matematicas, XIX(1–2):31–42, 1985.

PAC (RM_3) – Extension of LP by Implication

(Avron, Batens, da Costa, D'Ottaviano)

$$\text{PAC} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\} \rangle$$

$$a \mathbin{\tilde{\supset}} b \;=\; \begin{cases} b & a \neq f, \\ t & a = f. \end{cases}$$

This is an implication ($\mathcal{S}, \varphi \vdash \psi$ iff $\mathcal{S} \vdash \varphi \supset \psi$) for each 3-valued matrix: with $\mathcal{D} = \{t\}$ (Słupecki) or with $\mathcal{D} = \{t, \top\}$ (da Costa & D'Ottaviano).

A. Avron. *On an implication connective of RM*. Notre Dame Journal of Formal Logic, 27:201–209, 1986.

D. Batens. *Paraconsistent extensional propositional logics*. Logique et Analyse, 90–91:195–234, 1980.

I. D'Ottaviano. *The completeness and compactness of a three-valued first-order logic*. Revista Colombiana de Matematicas, XIX(1–2):31–42, 1985.

PAC (RM_3) – Extension of LP by Implication

(Avron, Batens, da Costa, D'Ottaviano)

$$\text{PAC} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\} \rangle$$

$$a \mathbin{\tilde{\supset}} b \;=\; \begin{cases} b & a \neq f, \\ t & a = f. \end{cases}$$

This is an implication ($\mathcal{S}, \varphi \vdash \psi$ iff $\mathcal{S} \vdash \varphi \supset \psi$) for each 3-valued matrix: with $\mathcal{D} = \{t\}$ (Słupecki) or with $\mathcal{D} = \{t, \top\}$ (da Costa & D'Ottaviano).

- Paraconsistent
- Maximal in both senses
- Normal

A. Avron. *On an implication connective of RM*. Notre Dame Journal of Formal Logic, 27:201–209, 1986.

D. Batens. *Paraconsistent extensional propositional logics*. Logique et Analyse, 90–91:195–234, 1980.

I. D'Ottaviano. *The completeness and compactness of a three-valued first-order logic*. Revista Colombiana de Matematicas, XIX(1–2):31–42, 1985.

Extensions of PAC by Propositional Constants

- $\mathsf{J}_3 = \mathsf{PAC}^{\mathsf{F}} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, \mathsf{F}\}\rangle$
- $\mathsf{PAC}^{\mathsf{B}} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, \mathsf{B}\}\rangle$
- $\mathsf{J}_3^{\mathsf{B}} = \mathsf{PAC}^{\mathsf{F},\mathsf{B}} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, \mathsf{F}, \mathsf{B}\}\rangle$

Extensions of PAC by Propositional Constants

- $J_3 = PAC^F = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, F\}\rangle$
- $PAC^B = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, B\}\rangle$
- $J_3^B = PAC^{F,B} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, F, B\}\rangle$

- PAC, J_3, PAC^B and J_3^B are all normal
- PAC, J_3, PAC^B and J_3^B are paraconsistent logic
- PAC and J_3 are maximal in both senses
(This is false for PAC^B and J_3^B, which are not $\neg$-contained in CL)

Notable 3-valued Paraconsistent Logics

Furthermore:

- J_3^B is the strongest 3-valued paraconsistent logic (every 3-valued paraconsistent logic can be embedded in it).*
- J_3 is the strongest 3-valued paraconsistent logic which is $\neg$-contained in CL.
- PAC^B is the strongest 3-valued non-exploding paraconsistent logic.**
- PAC is the strongest 3-valued paraconsistent logic which is both $\neg$-contained in CL and non-exploding.

* Sette's negation is not represented in PAC^B, but it is represented in J_3.

** That is: For every $\mathcal{S}$ such that Atoms($\mathcal{S}$) $\neq$ Atoms($\mathcal{L}$), there is a formula ψ such that $\mathcal{S} \nvdash_{PAC^B} \psi$.

Tweety Dilemma

$$\mathcal{S} = \left\{ \begin{array}{l} \mathit{bird}(\mathit{Tweety}) \mapsto \mathit{fly}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \supset \mathit{bird}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \supset \neg \mathit{fly}(\mathit{Tweety}) \\ \mathit{bird}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \end{array} \right\}$$

Tweety Dilemma

$$\mathcal{S} = \left\{ \begin{array}{l} \mathit{bird}(\mathit{Tweety}) \mapsto \mathit{fly}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \supset \mathit{bird}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \supset \neg \mathit{fly}(\mathit{Tweety}) \\ \mathit{bird}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \end{array} \right\}$$

$\mathcal{S}$ has six PAC-models:

	bird	*fly*	*penguin*
ν_1	$\top$	$\top$	$\top$
ν_2	$\top$	$\top$	*t*
ν_3	$\top$	*f*	$\top$

	bird	*fly*	*penguin*
ν_4	$\top$	*f*	*t*
ν_5	*t*	$\top$	$\top$
ν_6	*t*	$\top$	*t*

Tweety Dilemma

$$\mathcal{S} = \left\{ \begin{array}{l} \mathit{bird}(\mathit{Tweety}) \mapsto \mathit{fly}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \supset \mathit{bird}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \supset \neg \mathit{fly}(\mathit{Tweety}) \\ \mathit{bird}(\mathit{Tweety}) \\ \mathit{penguin}(\mathit{Tweety}) \end{array} \right\}$$

$\mathcal{S}$ has six PAC-models:

	bird	*fly*	*penguin*
ν_1	$\top$	$\top$	$\top$
ν_2	$\top$	$\top$	*t*
ν_3	$\top$	*f*	$\top$

	bird	*fly*	*penguin*
ν_4	$\top$	*f*	*t*
ν_5	*t*	$\top$	$\top$
ν_6	*t*	$\top$	*t*

Thus, $\mathcal{S}$ is classically inconsistent, yet:
$\mathcal{S} \vdash_{\mathsf{PAC}} \mathit{bird}(\mathit{Tweety})$, $\mathcal{S} \vdash_{\mathsf{PAC}} \mathit{penguin}(\mathit{Tweety})$, $\mathcal{S} \vdash_{\mathsf{PAC}} \neg\mathit{fly}(\mathit{Tweety})$,
while the negated assertions *cannot* be concluded.

Tweety Dilemma (Cont'd.)

Inconsistency is 'localized':

$$\mathcal{S}' = \mathcal{S} \bigcup \left\{ \begin{array}{l} \textit{elephant}(\textit{Fred}) \supset \neg \textit{fly}(\textit{Fred}) \\ \textit{elephant}(\textit{Fred}) \end{array} \right\}$$

Tweety Dilemma (Cont'd.)

Inconsistency is 'localized':

$$\mathcal{S}' = \mathcal{S} \bigcup \left\{ \begin{array}{l} \mathit{elephant}(\mathit{Fred}) \supset \neg \mathit{fly}(\mathit{Fred}) \\ \mathit{elephant}(\mathit{Fred}) \end{array} \right\}$$

$$\begin{array}{ll}
\mathcal{S}' \vdash_{\mathsf{PAC}} \mathit{bird}(\mathit{Tweety}) & \mathcal{S}' \nvdash_{\mathsf{PAC}} \neg \mathit{bird}(\mathit{Tweety}) \\
\mathcal{S}' \vdash_{\mathsf{PAC}} \mathit{penguin}(\mathit{Tweety}) & \mathcal{S}' \nvdash_{\mathsf{PAC}} \neg \mathit{penguin}(\mathit{Tweety}) \\
\mathcal{S}' \vdash_{\mathsf{PAC}} \neg \mathit{fly}(\mathit{Tweety}) & \mathcal{S}' \nvdash_{\mathsf{PAC}} \mathit{fly}(\mathit{Tweety}) \\
\mathcal{S}' \vdash_{\mathsf{PAC}} \mathit{elephant}(\mathit{Fred}) & \mathcal{S}' \nvdash_{\mathsf{PAC}} \neg \mathit{elephant}(\mathit{Fred}) \\
\mathcal{S}' \vdash_{\mathsf{PAC}} \neg \mathit{fly}(\mathit{Fred}) & \mathcal{S}' \nvdash_{\mathsf{PAC}} \mathit{fly}(\mathit{Fred})
\end{array}$$

Logics of Formal Inconsistency (LFIs)

(Carnielli, Coniglio, Marcos)

$$\mathrm{LFI} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\circ}, \tilde{\neg}\} \rangle$$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$t \wr \top$
f	f	f	f
$\top$	$t \wr \top$	f	$t \wr \top$

$\tilde{\vee}$	t	f	$\top$
t	t	t	$t \wr \top$
f	t	f	$t \wr \top$
$\top$	$t \wr \top$	$t \wr \top$	$t \wr \top$

$\tilde{\supset}$	t	f	$\top$
t	t	f	$t \wr \top$
f	t	t	$t \wr \top$
$\top$	$t \wr \top$	f	$t \wr \top$

	$\tilde{\neg}$
t	f
f	t
$\top$	$t \wr \top$

	$\tilde{\circ}$
t	t
f	t
$\top$	f

- 2 possible interpretations for $\neg$, 2^3 for $\wedge$, 2^5 for $\vee$, and 2^4 for $\supset$.
- Altogether 2^{13} (8Kb) distinct (paraconsistent and maximal) logics.

W. A. Carnielli, M. E. Coniglio, and J. Marcos. *Logics of formal inconsistency*. Handbook of Philosophical Logic 14, pp.1–95, 2007.

Logics of Formal Inconsistency (LFIs)

- There are the 2^{13} (8Kb) three-valued paraconsistent matrices $\mathcal{M}$ for the language of $\{\neg, \wedge, \vee, \supset\}$ such that $\mathbf{L}_{\mathcal{M}}$ is $\neg$-contained in classical logic and normal, with the connectives $\wedge$, $\vee$ and $\supset$ being conjunction, disjunction and implication (respectively).

Logics of Formal Inconsistency (LFIs)

- There are the 2^{13} (8Kb) three-valued paraconsistent matrices $\mathcal{M}$ for the language of $\{\neg, \wedge, \vee, \supset\}$ such that $\mathbf{L}_{\mathcal{M}}$ is $\neg$-contained in classical logic and normal, with the connectives $\wedge$, $\vee$ and $\supset$ being conjunction, disjunction and implication (respectively).
- The logics induced by the matrices in 8Kb are all distinct: different matrices in 8Kb induce different logics.

Logics of Formal Inconsistency (LFIs)

- There are the 2^{13} (8Kb) three-valued paraconsistent matrices $\mathcal{M}$ for the language of $\{\neg, \wedge, \vee, \supset\}$ such that $\mathbf{L}_{\mathcal{M}}$ is $\neg$-contained in classical logic and normal, with the connectives $\wedge$, $\vee$ and $\supset$ being conjunction, disjunction and implication (respectively).
- The logics induced by the matrices in 8Kb are all distinct: different matrices in 8Kb induce different logics.
- All the logics are paraconsistent and maximal in both senses.

Logics of Formal Inconsistency (LFIs)

- There are the 2^{13} (8Kb) three-valued paraconsistent matrices $\mathcal{M}$ for the language of $\{\neg, \wedge, \vee, \supset\}$ such that $\mathbf{L}_{\mathcal{M}}$ is $\neg$-contained in classical logic and normal, with the connectives $\wedge$, $\vee$ and $\supset$ being conjunction, disjunction and implication (respectively).
- The logics induced by the matrices in 8Kb are all distinct: different matrices in 8Kb induce different logics.
- All the logics are paraconsistent and maximal in both senses.
- All of the logics are embedded in J_3, which may defined as a logic in the language of $\{\neg, \wedge, \vee, \supset, \circ\}$ (i.e., where $\circ$ replaces F). Indeed, this makes no difference since:
  - $\tilde{\circ}(a) = (a \mathbin{\tilde{\wedge}} \tilde{\neg} a) \mathbin{\tilde{\supset}} f$, and
  - $f = \tilde{\circ}(a) \mathbin{\tilde{\wedge}} \tilde{\neg}\tilde{\circ}(a)$.

  This logic is also known as LFI1 (Carnielli, Coniglio, Marcos).

Many Other 3-valued Paraconsistent Logics...

This survey is by no means exhaustive. Other known logics include:

- Special cases of LFIs:
  - The logic SRMI, incorporating Sobociński's implication:

$$a \tilde{\to} b \;=\; \begin{cases} \top & \text{if } a = b = \top, \\ f & \text{if } a >_{\mathrm{t}} b \;\; (\text{where } t >_{\mathrm{t}} \top >_{\mathrm{t}} f), \\ t & \text{otherwise.} \end{cases}$$

    B. Sobociński. *Axiomatization of a partial system of three-value calculus of propositions*. Journal of Computing Systems 1, pp.23–55, 1952

  - Avron & Béziau logic PE3.

    A. Avron, J. Y. Béziau. *Self-extensional three-valued paraconsistent logics have no implication*. Logic Journal of the IGPL 25, pp.183–194, 2017

- Paraconsistent 3-valued logics with a single designated value.
  - M. Osorio, J. L. Carballido. *Brief study of G_3' logic*. Applied Non-Classical Logics 18, pp.475-499, 2008.
  - G. Robles, J. M. Mendéz. *A paraconsistent 3-valued logic related to Gödel logic G3*. Logic Journal of the IGPL 22, pp.515–538, 2014.

Plan of Module 1

1. Preliminaries; Multi-Valued Logics
2. Negation and Paraconsistency
3. Maximal Properties of Paraconsistent Logics
   - Maximal Paraconsistency
   - Maximality Relative to CL
4. The Simplest Paraconsistent Multi-Valued Logics (3-Valued Paraconsistent Logics)
5. **Combining Paraconsistency and Paracompleteness (4-Valued Paradefinite Logics)**
6. A General Construction
7. Other Approaches to Paraconsistency

Combining Paraconsistency and Paracompleteness

All the 3-values paraconsistent logic are ***complete***:

if $\mathcal{S}, \psi \vdash \varphi$ and $\mathcal{S}, \neg\psi \vdash \varphi$ then $\mathcal{S} \vdash \varphi$.

Combining Paraconsistency and Paracompleteness

All the 3-values paraconsistent logic are *complete*:

if $\mathcal{S}, \psi \vdash \varphi$ and $\mathcal{S}, \neg\psi \vdash \varphi$ then $\mathcal{S} \vdash \varphi$.

We would like to consider paraconsistent logics that are also able to handle incomplete data by *rejecting the law of excluded middle*, according to which any proposition is either 'true' (i.e., known) or its negation is 'true'.

Combining Paraconsistency and Paracompleteness

All the 3-values paraconsistent logic are *complete*:

$$\text{if } \mathcal{S}, \psi \vdash \varphi \text{ and } \mathcal{S}, \neg\psi \vdash \varphi \text{ then } \mathcal{S} \vdash \varphi.$$

We would like to consider paraconsistent logics that are also able to handle incomplete data by *rejecting the law of excluded middle*, according to which any proposition is either 'true' (i.e., known) or its negation is 'true'.

Definition

A logic $\mathfrak{L}$ is called *$\neg$-paracomplete*, if $\neg$ is a negation for $\mathfrak{L}$ which is not complete for $\mathfrak{L}$.

Combining Paraconsistency and Paracompleteness

All the 3-values paraconsistent logic are *complete*:

if $\mathcal{S}, \psi \vdash \varphi$ and $\mathcal{S}, \neg\psi \vdash \varphi$ then $\mathcal{S} \vdash \varphi$.

We would like to consider paraconsistent logics that are also able to handle incomplete data by *rejecting the law of excluded middle*, according to which any proposition is either ‘true’ (i.e., known) or its negation is ‘true’.

Definition

A logic $\mathfrak{L}$ is called *$\neg$-paracomplete*, if $\neg$ is a negation for $\mathfrak{L}$ which is not complete for $\mathfrak{L}$.

Definition

A logic $\mathfrak{L}$ is called *$\neg$-paradefinite* (‘beyond the definite’), if it is both $\neg$-paraconsistent and $\neg$-paracomplete.

Four-Valued Paradefinite Logics

The simplest semantic framework for combining paraconsistent and paracomplete reasoning.

Four-Valued Paradefinite Logics

The simplest semantic framework for combining paraconsistent and paracomplete reasoning.

Proposition

If $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a $\neg$-paradefinite matrix then there are four elements $t, f, \top$, and $\bot$ in $\mathcal{V}$ such that:

(1): $t \in \mathcal{D}$ and $\tilde{\neg} t \notin \mathcal{D}$, *(2): $f \notin \mathcal{D}$ and $\tilde{\neg} f \in \mathcal{D}$,*
(3): $\top \in \mathcal{D}$ and $\tilde{\neg} \top \in \mathcal{D}$, *(4): $\bot \notin \mathcal{D}$ and $\tilde{\neg} \bot \notin \mathcal{D}$,* *(5): $\tilde{\neg} t = f$.*

Proposition

Let $\mathcal{M}$ be a $\neg$-paradefinite four-valued matrix. Then $\mathcal{M}$ is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, in which:

$\mathcal{V} = \{t, f, \top, \bot\}$, $\mathcal{D} = \{t, \top\}$, *and*
$\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top \in \{t, \top\}$, $\tilde{\neg} \bot \in \{f, \bot\}$.

Negations in Four-Valued Paradefinite Logics

The last proposition leaves 4 choices for $\neg$.

One of them (Dunn-Belnap negation) is more natural:

Negations in Four-Valued Paradefinite Logics

The last proposition leaves 4 choices for $\neg$.

One of them (Dunn-Belnap negation) is more natural:

<u>Lemma</u>: Let $\mathcal{M}$ be a $\neg$-paradefinite four-valued matrix. Then:

- If $\neg$ is left involutive for $\mathfrak{L}_{\mathcal{M}}$ (i.e., $\neg\neg p \vdash_{\mathfrak{L}_{\mathcal{M}}} p$) then $\tilde{\neg}\bot = \bot$.
- If $\neg$ is right involutive for $\mathfrak{L}_{\mathcal{M}}$ (i.e., $p \vdash_{\mathfrak{L}_{\mathcal{M}}} \neg\neg p$) then $\tilde{\neg}\top = \top$.

Negations in Four-Valued Paradefinite Logics

The last proposition leaves 4 choices for $\neg$.

One of them (Dunn-Belnap negation) is more natural:

Lemma: Let $\mathcal{M}$ be a $\neg$-paradefinite four-valued matrix. Then:

- If $\neg$ is left involutive for $\mathfrak{L}_{\mathcal{M}}$ (i.e., $\neg\neg p \vdash_{\mathfrak{L}_{\mathcal{M}}} p$) then $\tilde{\neg}\bot = \bot$.
- If $\neg$ is right involutive for $\mathfrak{L}_{\mathcal{M}}$ (i.e., $p \vdash_{\mathfrak{L}_{\mathcal{M}}} \neg\neg p$) then $\tilde{\neg}\top = \top$.

Proof:

$\neg\neg p \vdash_{\mathfrak{L}_{\mathcal{M}}} p$, thus $\tilde{\neg}\bot \neq f$ (otherwise $\nu(p) = \bot$ is a countermodel).
Since $\tilde{\neg}\bot \in \{f, \bot\}$, we have that $\tilde{\neg}\bot = \bot$.

$p \vdash_{\mathfrak{L}_{\mathcal{M}}} \neg\neg p$, thus $\tilde{\neg}\top \neq t$ (otherwise $\nu(p) = \top$ is a countermodel).
Since $\tilde{\neg}\top \in \{t, \top\}$, we have that $\tilde{\neg}\top = \top$. $\Box$

Case Study: Dunn-Belnap Logic (FDE)

Motivation – Integration of Information Sources

A processor collects and processes information from a set of sources. Each source may provide the processor with information about *atomic* formulas. The information has the form of a truth-value in $\{1, 0, ?\}$.

The processor assigns to an atom p a subset $d(p)$ of $\{0, 1\}$:

- $1 \in d(p)$ iff some source has assigned 1 to p
- $0 \in d(p)$ iff some source has assigned 0 to p

Case Study: Dunn-Belnap Logic (FDE)

Motivation – Integration of Information Sources

A processor collects and processes information from a set of sources. Each source may provide the processor with information about *atomic* formulas. The information has the form of a truth-value in $\{1, 0, ?\}$.

The processor assigns to an atom p a subset $d(p)$ of $\{0, 1\}$:

- $1 \in d(p)$ iff some source has assigned 1 to p
- $0 \in d(p)$ iff some source has assigned 0 to p

$d(p) = \{1\}$	p is known to be true and not known to be false
$d(p) = \{0\}$	p is known to be false and not known to be true
$d(p) = \{0, 1\}$	p is known to be true and known to be false
$d(p) = \emptyset$	p is not known to be true and not knwon to be false

Dunn-Belnap Logic (Cont'd.)

The processor's valuation is extended to $\{\neg, \vee, \wedge\}$ as follows:

(db1) $0 \in d(\neg\varphi)$ iff $1 \in d(\varphi)$

(db2) $1 \in d(\neg\varphi)$ iff $0 \in d(\varphi)$

(db3) $1 \in d(\varphi \vee \psi)$ iff $1 \in d(\varphi)$ or $1 \in d(\psi)$

(db4) $0 \in d(\varphi \vee \psi)$ iff $0 \in d(\varphi)$ and $0 \in d(\psi)$

(db5) $1 \in d(\varphi \wedge \psi)$ iff $1 \in d(\varphi)$ and $1 \in d(\psi)$

(db6) $0 \in d(\varphi \wedge \psi)$ iff $0 \in d(\varphi)$ or $0 \in d(\psi)$.

Dunn-Belnap Logic (Cont'd.)

The processor's valuation is extended to $\{\neg, \vee, \wedge\}$ as follows:

(db1) $0 \in d(\neg\varphi)$ iff $1 \in d(\varphi)$

(db2) $1 \in d(\neg\varphi)$ iff $0 \in d(\varphi)$

(db3) $1 \in d(\varphi \vee \psi)$ iff $1 \in d(\varphi)$ or $1 \in d(\psi)$

(db4) $0 \in d(\varphi \vee \psi)$ iff $0 \in d(\varphi)$ and $0 \in d(\psi)$

(db5) $1 \in d(\varphi \wedge \psi)$ iff $1 \in d(\varphi)$ and $1 \in d(\psi)$

(db6) $0 \in d(\varphi \wedge \psi)$ iff $0 \in d(\varphi)$ or $0 \in d(\psi)$.

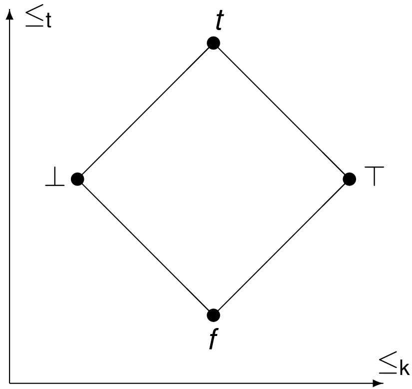
The corresponding interpretations of the connectives:

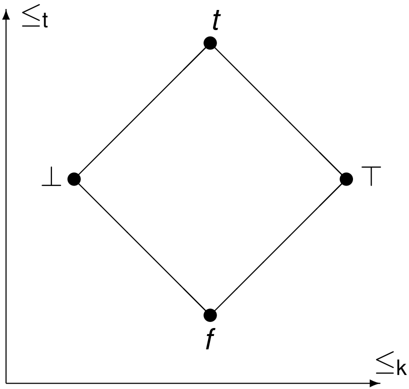
$\tilde{\vee}$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\tilde{\wedge}$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

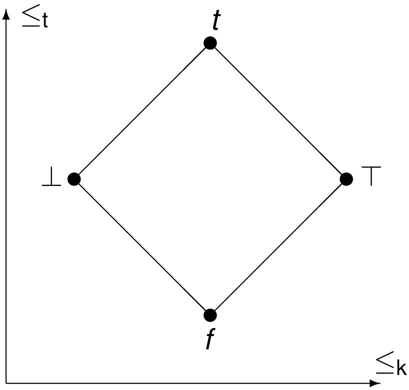
Belnap's Bilattice $\mathcal{FOUR}$





FDE $= \langle \mathcal{V}, \mathcal{D}, \{\tilde{\neg}, \tilde{\vee}, \tilde{\wedge}\}\rangle$: Dunn-Belnap matrix for $\{\neg, \vee, \wedge\}$,
where $\mathcal{V} = \{t, f, \top, \bot\}$ and $\mathcal{D} = \{t, \top\}$.

Belnap's Bilattice $\mathcal{FOUR}$



FDE $= \langle \mathcal{V}, \mathcal{D}, \{\tilde{\neg}, \tilde{\vee}, \tilde{\wedge}\}\rangle$: Dunn-Belnap matrix for $\{\neg, \vee, \wedge\}$, where $\mathcal{V} = \{t, f, \top, \bot\}$ and $\mathcal{D} = \{t, \top\}$.

The logic induced by FDE is semi-normal, paradefinite, and $\neg$-contained in CL.

Dunn-Belnap Logic (Cont'd.)

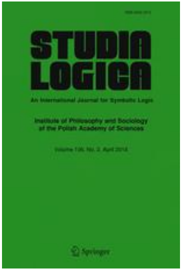
N. D. Belnap. *How a computer should think*. In: Contemporary Aspects of Philosophy, pages 30–56. G. Ryle, editor, Oriel Press, 1977.

N. D. Belnap. *A useful four-valued logic*. In: Modern Uses of Multiple-Valued Logics, pages 7–37. J. M. Dunn and G. Epstein, editors, Reidel Publishing Company, 1977.

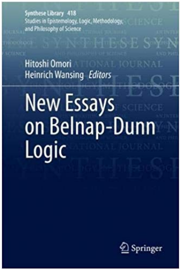
Dunn-Belnap Logic (Cont'd.)

N. D. Belnap. *How a computer should think*. In: Contemporary Aspects of Philosophy, pages 30–56. G. Ryle, editor, Oriel Press, 1977.

N. D. Belnap. *A useful four-valued logic*. In: Modern Uses of Multiple-Valued Logics, pages 7–37. J. M. Dunn and G. Epstein, editors, Reidel Publishing Company, 1977.



(a) Studia Logica 105(6), Omori & Wansing, editors, December 2017



(b) Synthese Library 418, Omori & Wansing, editors, Springer, December 2019

Expansions of FDE

- FDE is only semi-normal (no implication is definable in it)
- FDE is not maximal in either senses (LP extends it by $\psi \vee \neg\psi$)
- There are no FDE-tautologies (if $\forall p \in \text{Atoms}(\psi)\ \nu(p) = \bot$ then $\nu(\psi) = \bot$)

Expansions of FDE

- FDE is only semi-normal (no implication is definable in it)
- FDE is not maximal in either senses (LP extends it by $\psi \vee \neg\psi$)
- There are no FDE-tautologies (if $\forall p \in \text{Atoms}(\psi)\ \nu(p) = \bot$ then $\nu(\psi) = \bot$)

Other Useful Connectives

- An $\mathcal{M}$-implication for every paradefinite four-valued matrix $\mathcal{M}$

$$a \tilde{\supset} b = \begin{cases} b & \text{if } a \in \{t, \top\}, \\ t & \text{if } a \in \{f, \bot\}. \end{cases}$$

- Using the $\leq_k$-order: $-$ ($\leq_k$-reversing), $\oplus$ ($\leq_k$-lub), $\otimes$ ($\leq_k$-glb)

$\tilde{\oplus}$	t	f	$\top$	$\bot$
t	t	$\top$	$\top$	t
f	$\top$	f	$\top$	f
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	t	f	$\top$	$\bot$

$\tilde{\otimes}$	t	f	$\top$	$\bot$
t	t	$\bot$	t	$\bot$
f	$\bot$	f	f	$\bot$
$\top$	t	f	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\tilde{-}$	
t	t
f	f
$\top$	$\bot$
$\bot$	$\top$

- Constants: T (truth), F (falsity), B (both) and N (neither).

The Full Language and its Matrix

$\mathcal{L}_{All} = \{\neg, \vee, -, \wedge, \oplus, \otimes, \supset, \mathsf{F}, \mathsf{T}, \mathsf{B}, \mathsf{N}\}$ – the full language.

4All – the expansion of FDE to $\mathcal{L}_{All}$.

1

A Maximal Expansion of FDE

The Full Language and its Matrix

$\mathcal{L}_{All} = \{\neg, \vee, -, \wedge, \oplus, \otimes, \supset, \mathsf{F}, \mathsf{T}, \mathsf{B}, \mathsf{N}\}$ – the full language.

4All – the expansion of FDE to $\mathcal{L}_{All}$.

Not all the connectives in $\mathcal{L}_{All}$ are really necessary. For instance, the following languages are *functionally complete* for $\{t, f, \top, \bot\}$:[1]

- The language of $\{\neg, \vee, \wedge, \supset, \mathsf{B}, \mathsf{N}\}$
- The language of $\{\neg, \vee, \wedge, \supset, \oplus, \otimes, \mathsf{F}\}$

[1] Every function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in the language.

A Maximal Expansion of FDE

The Full Language and its Matrix

$\mathcal{L}_{All} = \{\neg, \vee, -, \wedge, \oplus, \otimes, \supset, \mathsf{F}, \mathsf{T}, \mathsf{B}, \mathsf{N}\}$ – the full language.

4All – the expansion of FDE to $\mathcal{L}_{All}$.

Not all the connectives in $\mathcal{L}_{All}$ are really necessary. For instance, the following languages are *functionally complete* for $\{t, f, \top, \bot\}$:[1]

- The language of $\{\neg, \vee, \wedge, \supset, \mathsf{B}, \mathsf{N}\}$
- The language of $\{\neg, \vee, \wedge, \supset, \oplus, \otimes, \mathsf{F}\}$

The logic that is induced by 4All is:

- normal
- paradefinite
- maximally paraconsistent

[1] Every function $g: \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in the language.

A Maximal Monotonic Expansion

(Following Belnap's intuition on the sources integration system)

$\mathcal{L}_{Mon} = \{\neg, \vee, \wedge, \mathsf{B}, \mathsf{N}\}$

4Mon – the expansion of FDE to $\mathcal{L}_{Mon}$.

A Maximal Monotonic Expansion

(Following Belnap's intuition on the sources integration system)

$\mathcal{L}_{Mon} = \{\neg, \vee, \wedge, \mathsf{B}, \mathsf{N}\}$

4Mon – the expansion of FDE to $\mathcal{L}_{Mon}$.

Expressive power: A function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\mathcal{L}_{Mon}$ iff it is $\leq_k$-monotonic (i.e., if $\forall i \; a_i \leq_k b_i$ then $g(a_1, \ldots, a_n) \leq_k g(b_1, \ldots, b_n)$).

Thus: the logic that is induced by 4Mon contains every logic that is induced by a 4-valued paradefinite matrix that employs only $\leq_k$-monotonic functions.

A Maximal Monotonic Expansion

(Following Belnap's intuition on the sources integration system)

$\mathcal{L}_{Mon} = \{\neg, \vee, \wedge, \mathsf{B}, \mathsf{N}\}$

4Mon – the expansion of FDE to $\mathcal{L}_{Mon}$.

Expressive power: A function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\mathcal{L}_{Mon}$ iff it is $\leq_k$-monotonic (i.e., if $\forall i\ a_i \leq_k b_i$ then $g(a_1, \ldots, a_n) \leq_k g(b_1, \ldots, b_n)$).

Thus: the logic that is induced by 4Mon contains every logic that is induced by a 4-valued paradefinite matrix that employs only $\leq_k$-monotonic functions.

The logic that is induced by 4Mon is:

- semi-normal
- paradefinite
- maximally paraconsistent

A Maximal Classical Expansion

$\mathcal{L}_{CC} = \{\neg, -, \vee, \wedge, \supset\}$

4CC – the expansion of FDE to $\mathcal{L}_{CC}$.

A Maximal Classical Expansion

$\mathcal{L}_{CC} = \{\neg, -, \vee, \wedge, \supset\}$
4CC – the expansion of FDE to $\mathcal{L}_{CC}$.

Expressive power: $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\mathcal{L}_{CC}$ iff it is $\{t, f\}$-closed. (if $\forall p \in \text{Atoms}(\psi)\ \nu(p) \in \{t, f\}$ then $\nu(\psi) \in \{t, f\}$)

Thus: the logic that is induced by 4CC contains every logic that is induced by a 4-valued paradefinite matrix and is $\neg$-contained in CL.

A Maximal Classical Expansion

$\mathcal{L}_{CC} = \{\neg, -, \vee, \wedge, \supset\}$
4CC – the expansion of FDE to $\mathcal{L}_{CC}$.

Expressive power: $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\mathcal{L}_{CC}$ iff it is $\{t, f\}$-closed. (if $\forall p \in \text{Atoms}(\psi)\ \nu(p) \in \{t, f\}$ then $\nu(\psi) \in \{t, f\}$)

Thus: the logic that is induced by 4CC contains every logic that is induced by a 4-valued paradefinite matrix and is $\neg$-contained in CL.

The logic that is induced by 4CC is:

- normal
- paradefinite
- $\neg$-contained in CL
- maximal in both senses

Plan of Module 1

1. Preliminaries; Multi-Valued Logics
2. Negation and Paraconsistency
3. Maximal Properties of Paraconsistent Logics
    - Maximal Paraconsistency
    - Maximality Relative to CL
4. The Simplest Paraconsistent Multi-Valued Logics (3-Valued Paraconsistent Logics)
5. Combining Paraconsistency and Paracompleteness (4-Valued Paradefinite Logics)
6. **A General Construction**
7. Other Approaches to Paraconsistency

A General Construction

Proposition

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an n-valued matrix ($n > 3$) for a language $\mathcal{L}$ containing two unary connectives $\neg$ and $\diamond$, and a binary connective $\supset$.

1. *Suppose that the following conditions hold in $\mathcal{M}$:*
   - $\mathcal{V} = \{t, f, \top, \bot_1, \ldots, \bot_{n-3}\}$ *and* $\mathcal{D} = \{t, \top\}$*;*
   - $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, *and* $\tilde{\neg} x = x$ *otherwise;*
   - $\tilde{\diamond} t = f$, $\tilde{\diamond} f = t$, $\tilde{\diamond} \top = \bot_1$, $\tilde{\diamond} \bot_i = \bot_{i+1}$ *for* $i < n - 3$, *and* $\tilde{\diamond} \bot_{n-3} = \top$*;*
   - $\tilde{\supset}$ *is defined by* $a \tilde{\supset} b = t$ *if* $a \notin \mathcal{D}$, *and* $a \tilde{\supset} b = b$ *otherwise.*

   Then $\mathfrak{L}_{\mathcal{M}}$ is a normal, paradefinite, and maximally paraconsistent n-valued logic, which is not equivalent to any m-valued logic with $m < n$.

2. *Suppose that, in addition to the conditions above, every other connective of $\mathcal{M}$ is $\{t, f\}$-closed. Then $\mathfrak{L}_{\mathcal{M}}$ is maximal in both senses.*

Plan of Module 1

1. Preliminaries; Multi-Valued Logics
2. Negation and Paraconsistency
3. Maximal Properties of Paraconsistent Logics
    - Maximal Paraconsistency
    - Maximality Relative to CL
4. The Simplest Paraconsistent Multi-Valued Logics (3-Valued Paraconsistent Logics)
5. Combining Paraconsistency and Paracompleteness (4-Valued Paradefinite Logics)
6. A General Construction
7. **Other Approaches to Paraconsistency**

Different Approaches to Paraconsistency

- Multi-Valued Truth Functionality (algebraic methods, matrices)
- Possible Worlds (modalities, intuitionism)
- Relevance (variable sharing)
- Non-Determinism (relaxation of truth functionality, Nmatrices)

Different Approaches to Paraconsistency

- Multi-Valued Truth Functionality (algebraic methods, matrices)
- Possible Worlds (modalities, intuitionism)
- Relevance (variable sharing)
- Non-Determinism (relaxation of truth functionality, Nmatrices)