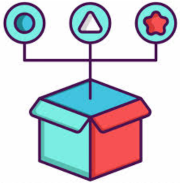
Module 3

Knowledge Representation & Rationality Postulates



Plan of Module 3

1. Knowledge Representation Considerations
   - Extended expressive power
     - ▷ Distinction between strict and defeasible premises
     - ▷ Extending arguments with priorities
     - ▷ Trading sequents by hypersequents
     - ▷ Introducing abducitve sequents
   - Consistency and minimality
   - Alternative formalizations of attack rules
2. General Properties
   - Reasoning with maximal consistency
   - Rationality postulates

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Strict and Defeasible Premises

- It is usual to divide the assumptions to two types: strict (non-attacked) ones $\mathcal{X}$ and defeasible ones $\mathcal{S}$.
- Attack rules are modified:

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Undercut$_{\mathcal{X}}$

$$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg\bigwedge\Gamma_2' \quad \neg\bigwedge\Gamma_2' \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2} \quad \Gamma_2' \neq \emptyset,\ \Gamma_2' \cap \mathcal{X} = \emptyset$$

Direct Undercut$_{\mathcal{X}}$

$$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg\gamma_2 \quad \neg\gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2} \quad \gamma_2 \notin \mathcal{X}$$

Consistency Undercut$_{\mathcal{X}}$

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Incorporation of Modalities, Revisited

$$\mathfrak{L} = \mathsf{S4},\ \mathcal{R} = \{\mathrm{DirDef}\},\ \mathcal{S} = \{p,\ q,\ p \supset \Box r,\ q \supset \Box \neg r\}$$

$A_1 = p \Rightarrow p$

$A_2 = q \Rightarrow q$

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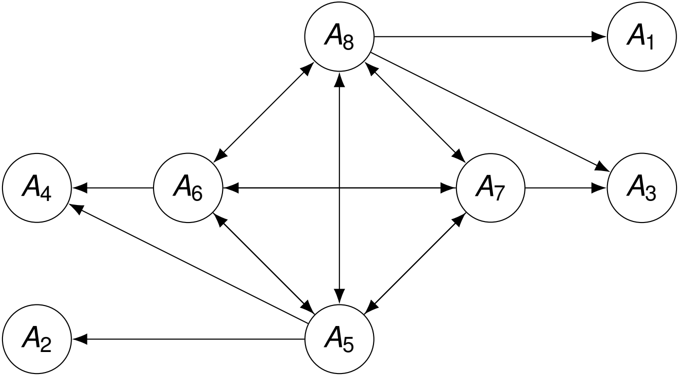

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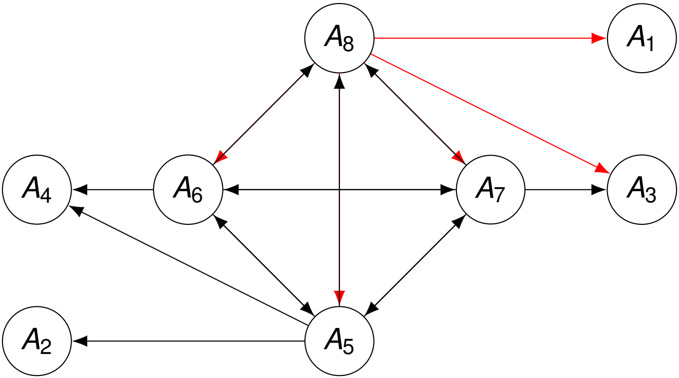

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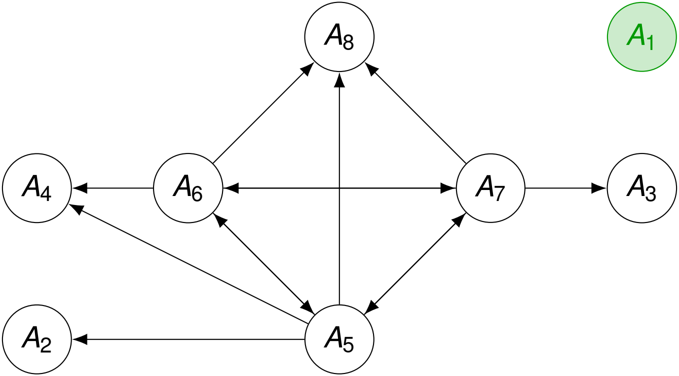

The attacks of A_8 (on A_1, A_3, A_5, A_6 and A_7) are removed.

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  - (Brewka's stratification by subset relation)
    $\Gamma_1 = \emptyset$, or
    $\exists i \in \mathbb{N}$ s.t. $\{\psi \in \Gamma_1 \mid \pi(\psi) = i\} \supsetneq \{\psi \in \Gamma_2 \mid \pi(\psi) = i\}$
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Prioritized (Sequent-based) AF

Defeat = Attack + preference considerations:

A_1 defeats A_2, if there is an attack rule $\mathcal{R}$ such that A_1 $\mathcal{R}$-attacks A_2, and A_2 is not $\prec_\pi$-preferred (i.e., not $\prec_\pi$-smaller) than A_1.

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A *prioritized (sequent-based) argumentation framework*[1] for $\mathcal{S}$, based on a logic $\mathfrak{L}$, a set $\mathcal{A}$ of attack rules, and a preference order $\preceq_\pi$, is the argumentation framework $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathsf{Dft}_{\preceq_\pi}(\mathcal{A})\rangle$, where:

- $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of the $\mathcal{S}$-based $\mathfrak{L}$-arguments, and
- $(A_1, A_2) \in \mathsf{Dft}_{\preceq_\pi}(\mathcal{A})$ iff A_1 $\mathcal{R}$-attacks A_2, for some $\mathcal{R} \in \mathcal{A}$ and $A_2 \not\prec_\pi A_1$.

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Dung semantics (and so the entailment relations) are defined as before, where attacks are replaced by defeats.

[1] O. Arieli, A. Borg and C. Straßer, *Prioritized Sequent-Based Argumentation*, AAMAS'2018, pages 1105–1113, ACM, 2018.

Example

$\mathfrak{L} = \mathsf{CL},\ \mathcal{R} = \{\mathrm{Ucut}\},\ \mathcal{S} = \{p, q, \neg p \vee \neg q\}$

$A_1 = p \Rightarrow p$ $\quad A_3 = \neg p \vee \neg q \Rightarrow \neg p \vee \neg q$ $\quad A_6 = p, q \Rightarrow p \wedge q$

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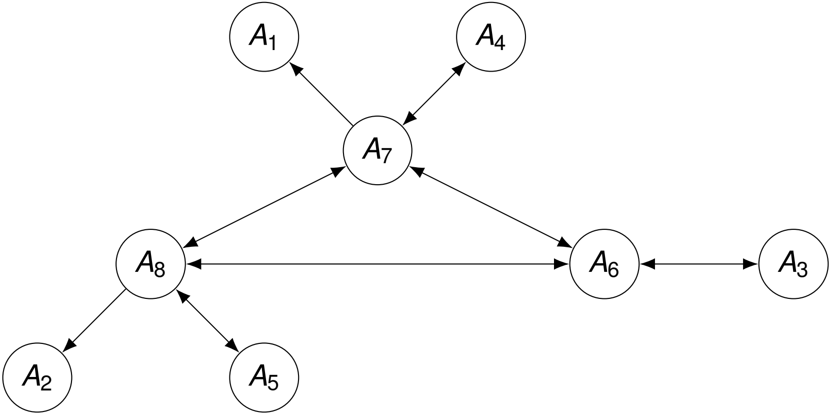

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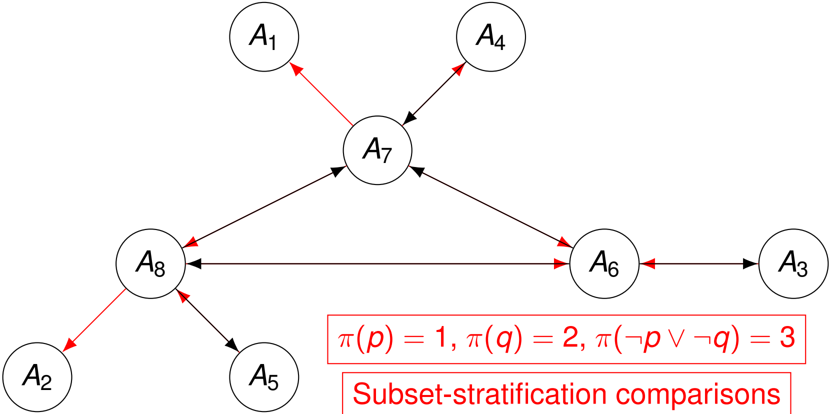

$\pi(p) = 1,\ \pi(q) = 2,\ \pi(\neg p \vee \neg q) = 3$

Subset-stratification comparisons

Example (Cont'd.)

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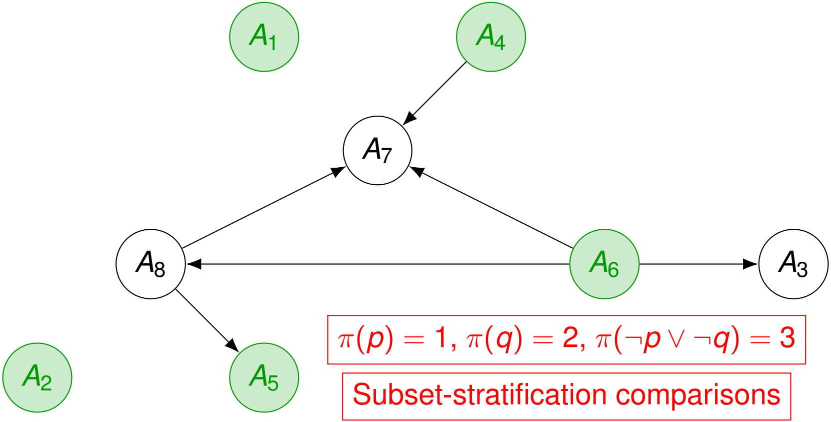

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Subset-stratification comparisons

Another Example

An flat owner negotiates the construction of a swimming pool (s), a tennis-court (t) and a private car-park (p) with his tenants. It is known that any investment in two or more of these facilities will increase the rent (r), otherwise the rent will not be changed. The tenants do not have a particular preference among these options, but if they have to make a choice, they prefer not to have two sport facilities (s and t) and definitely do not want to increase the rent. Based on these inputs, that flat owner needs to reach a recommendation about the facility (or facilities) to be constructed.

S. Konieczny, R. Pino Pérez. *Merging information under constraints: a logical framework*. Journal of Logic and Computation 5(12):773–808, 2002.

- The rent should not be increased: $\neg r$
- The rent increases if more than one facility is constructed: $\psi_1 = r \leftrightarrow ((s \wedge t) \vee (s \wedge p) \vee (t \wedge p))$
- No two sport facilities: $\psi_2 = s \supset \neg t$, $\psi_3 = t \supset \neg s$ (CL-equivalent)

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Representation by $\mathcal{AF}(\mathcal{S}) = \langle \text{Arg}_{\mathfrak{L}}(\mathcal{S}), \text{Dft}_{\preceq_\pi}(\mathcal{A}) \rangle$, where:

- $\mathfrak{L} = \text{CL}$, $\mathcal{S} = \{s, t, p, \neg r, \psi_1, \psi_2, \psi_3\}$,
- attacks by Ucut and ConUcut,
- $\pi(\neg r) = 1$, $\pi(\psi_1) = \pi(\psi_2) = \pi(\psi_3) = 2$, $\pi(s) = \pi(t) = \pi(p) = 3$, comparison by subset-stratifications.

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Observations:

- Argument of the form $x, y \Rightarrow x \wedge y$ for $x, y \in \{p, s, t\}$ are Ucut-defeated by $\neg r, \psi_1 \Rightarrow \neg(x \wedge y)$. ▷ **No two facilities.**
- $s \Rightarrow s$ and $t \Rightarrow t$ are respectively Ucut-defeated by $t, \psi_3 \Rightarrow \neg s$ and $s, \psi_2 \Rightarrow \neg t$. ▷ **No sport facility is built.**
- $p, \neg r, \psi_1, \psi_2, \psi_3 \Rightarrow p$ is defended since it is attacked only by arguments whose support set is classically inconsistent, and these attacks are counter ConUcut-defeated by the tautological argument $\Rightarrow \neg \bigwedge \mathcal{S}$. ▷ **Only a parking lot is built.**

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Extending Sequents to Hypersequents

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Motivation: For some central logics (including the modal logic S5, the relevance logic RM, and Gödel–Dummett logic LC) cut-free sequent calculi are not known.

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The logics S5, RM and LC do have *cut-free hyperseqeunt calculi*.

Definition (Mints, 1974, Pottinger 1983, Avron 1987)

A *hypersequent* is a finite multiset of sequents $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$

A. Avron, *A constructive analysis of RM*, Journal of Symbolic Logic 52(4), pages 939–951, 1987.

Construction of Arguments; Hypersequent Calculi

- Intuition: $\Gamma_1 \Rightarrow \psi_1 \mid \ldots \mid \Gamma_n \Rightarrow \psi_n$ is a ‘disjunction’ of the sequents $\Gamma_i \Rightarrow \psi_i$ ($i = 1, \ldots n$).

- Arguments = Derived hypersequent.

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- Arguments = Derived hypersequent.

- Inference rules:
    - Adaptation of logical sequent-based rules:

$$\frac{\mathcal{H}_1 \mid \Gamma, \psi \Rightarrow \Delta, \phi \mid \mathcal{H}_2}{\mathcal{H}_1 \mid \Gamma \Rightarrow \Delta, \psi \supset \phi \mid \mathcal{H}_2} \quad [\Rightarrow\supset]$$

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  - Additive and multiplicative forms:

$$\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \phi \mid \mathcal{H}' \qquad \mathcal{H} \mid \Gamma \Rightarrow \Delta, \psi \mid \mathcal{H}'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \phi \wedge \psi \mid \mathcal{H}'} \quad [\Rightarrow\wedge]$$

$$\frac{\mathcal{H}_1 \mid \Gamma_1 \Rightarrow \Delta_1, \phi \mid \mathcal{H}_1' \qquad \mathcal{H}_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi \mid \mathcal{H}_2'}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \phi \wedge \psi \mid \mathcal{H}_1' \mid \mathcal{H}_2'} \quad [\Rightarrow\wedge]$$

Construction of Arguments; Hypersequent Calculi

- Inference rules: (Cont'd.)
  - Internal-external structural rules:

$$\frac{\mathcal{H}_1 \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}_2}{\mathcal{H}_1 \mid \Gamma, \varphi \Rightarrow \Delta \mid \mathcal{H}_2} \quad \text{[Internal Weakening (L)]}$$

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Construction of Arguments; Hypersequent Calculi

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  - Additional structural hypersequent-based rules:

$$\frac{\mathcal{H}_1 \mid \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \mid \mathcal{H}_2}{\mathcal{H}_1 \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta' \mid \mathcal{H}_2} \quad \text{[Splitting]}$$

Example: HLK, Hypersequential Variation of LK

Axioms: $\psi \Rightarrow \psi$

Logical Rules:

$$[\neg\Rightarrow]\quad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \varphi}{\mathcal{H} \mid \neg\varphi, \Gamma \Rightarrow \Delta} \qquad [\Rightarrow\neg]\quad \frac{\mathcal{H} \mid \varphi, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \neg\varphi}$$

$$[\supset\Rightarrow]\quad \frac{\mathcal{H}_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \quad \mathcal{H}_2 \mid \psi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \Delta_1, \Delta_2} \qquad [\Rightarrow\supset]\quad \frac{\mathcal{H} \mid \Gamma, \varphi \Rightarrow \Delta, \psi}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$[\wedge\Rightarrow]\quad \frac{\mathcal{H} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad [\Rightarrow\wedge]\quad \frac{\mathcal{H}_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \quad \mathcal{H}_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \wedge \psi}$$

$$[\vee\Rightarrow]\quad \frac{\mathcal{H}_1 \mid \Gamma_1, \varphi \Rightarrow \Delta_1 \quad \mathcal{H}_2 \mid \Gamma_2, \psi \Rightarrow \Delta_2}{\mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_1, \Delta_2} \qquad [\Rightarrow\vee]\quad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \varphi, \psi}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Structural Rules:

$$[\mathrm{EC}]\quad \frac{\mathcal{H} \mid A \mid A}{\mathcal{H} \mid A} \qquad [\mathrm{EW}]\quad \frac{\mathcal{H}}{\mathcal{H} \mid A}$$

$$[\mathrm{IC}]\quad \frac{\mathcal{H} \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \varphi \Rightarrow \Delta} \quad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \varphi} \qquad [\mathrm{IW}]\quad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \varphi \Rightarrow \Delta} \quad \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \varphi}$$

$$[\mathrm{Sp}]\quad \frac{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2} \qquad [\mathrm{Cut}]\quad \frac{\mathcal{H} \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \quad \mathcal{H} \mid \varphi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

HS5, A (Cut-Free) Hypersequent Calculus for S5

HS5:

1. All rules of HLK, except for [Sp];
2. The following modality rules:

$$[\Box\Rightarrow] \quad \frac{\mathcal{H} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{H} \mid \Gamma, \Box\varphi \Rightarrow \Delta} \qquad [\Rightarrow\Box] \quad \frac{\mathcal{H} \mid \Box\Gamma \Rightarrow \varphi}{\mathcal{H} \mid \Box\Gamma \Rightarrow \Box\varphi}$$

$$[\mathrm{MS}] \quad \frac{\mathcal{H} \mid \Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta_1, \Delta_2}{\mathcal{H} \mid \Box\Gamma_1 \Rightarrow \Box\Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}$$

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Ax5: $\neg\Box\psi \supset \Box\neg\Box\psi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\psi \Rightarrow \Box\psi}{\Box\psi, \neg\Box\psi \Rightarrow}[\neg\Rightarrow]}{\Box\psi \Rightarrow \mid \neg\Box\psi \Rightarrow}[\mathrm{MS}]}{\Rightarrow \neg\Box\psi \mid \neg\Box\psi \Rightarrow}[\Rightarrow\neg]}{\Rightarrow \Box\neg\Box\psi \mid \neg\Box\psi \Rightarrow}[\Rightarrow\Box]}{\neg\Box\psi \Rightarrow \Box\neg\Box\psi \mid \neg\Box\psi \Rightarrow \Box\neg\Box\psi}[\mathrm{IW}\ \times 2]}{\neg\Box\psi \Rightarrow \Box\neg\Box\psi}[\mathrm{EC}]}{\Rightarrow \neg\Box\psi \supset \Box\neg\Box\psi}[\Rightarrow\supset]$$

Hypersequent-based Attack Rules

$\mathcal{H}_1 = \Gamma_1 \Rightarrow \psi_1 \mid \dots \mid \Gamma_n \Rightarrow \psi_n$

$\mathcal{H}_2 = \Delta_1, \Delta'_1 \Rightarrow \phi_1 \mid \dots \mid \Delta_j, \Delta'_j \Rightarrow \phi_j \mid \dots \mid \Delta_m, \Delta'_m \Rightarrow \phi_m$

$\overline{\mathcal{H}}$: elimination of $\mathcal{H}$

Undercut_H

$$\frac{\mathcal{H}_1 \qquad \bigvee_{i=1}^{n} \psi_i \Rightarrow \neg \bigwedge \Delta_j \qquad \neg \bigwedge \Delta_j \Rightarrow \bigvee_{i=1}^{n} \psi_i \qquad \mathcal{H}_2}{\overline{\mathcal{H}_2}} \; \text{Ucut}_H$$

Rebuttal_H

$$\frac{\mathcal{H}_1 \qquad \bigvee_{i=1}^{n} \psi_i \Rightarrow \neg \phi_j \qquad \neg \phi_j \Rightarrow \bigvee_{i=1}^{n} \psi_i \qquad \mathcal{H}_2}{\overline{\mathcal{H}_2}} \; \text{Reb}_H$$

Consistency Undercut_H

$$\frac{\Rightarrow \neg \bigwedge (\bigcup_{i=1}^{m} \Delta_i \cup \bigcup_{i=1}^{m} \Delta'_i) \qquad \mathcal{H}_2}{\overline{\mathcal{H}_2}} \; \text{ConUcut}_H$$

Hypersequent-based Argumentation Frameworks

A *hypersequent-based argumentation framework* for $\mathcal{S}$, based on a logic $\mathfrak{L}$ with a sound and complete hypersequent calculus $\mathcal{C}$ and a set $\mathcal{A}$ of hypersequent-based attack rules, is an argumentation framework of the form $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathsf{Attack}(\mathcal{A}) \rangle$, where:

- $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of the $\mathcal{C}$-derived $\mathcal{S}$-based $\mathfrak{L}$-hypersequents,
- $(\mathcal{H}_1, \mathcal{H}_2) \in \mathsf{Attack}(\mathcal{A})$ iff $\mathcal{H}_1$ $\mathcal{R}$-attacks $\mathcal{H}_2$, for some $\mathcal{R} \in \mathcal{A}$.

A. Borg, C. Straßer and O. Arieli, *A generalized proof-theoretic approach to logical argumentation based on hypersequents*, Studia Logica 109(1), pages 167–238, 2021.

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Benefits:

- Cut elimination is guaranteed for more settings
- Consistency of extensions' conclusions (see next example)

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Example, Revisited

$$\mathfrak{L} = \text{CL}, \ \mathcal{R} = \{\text{Def}\}, \ \mathcal{S} = \{p, q, \neg p \vee \neg q\}$$

$A_1 = p \Rightarrow p$ $\quad A_3 = \neg p \vee \neg q \Rightarrow \neg p \vee \neg q$ $\quad A_6 = p, q \Rightarrow p \wedge q$

$A_2 = q \Rightarrow q$ $\quad A_4 = p \Rightarrow \neg((\neg p \vee \neg q) \wedge q)$ $\quad A_7 = \neg p \vee \neg q, q \Rightarrow \neg p$

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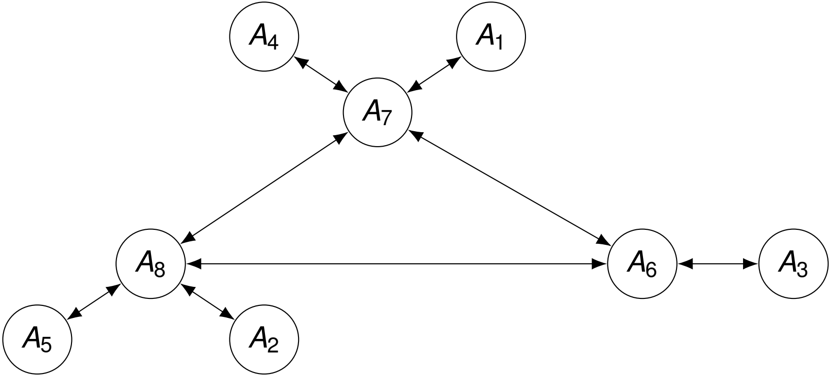

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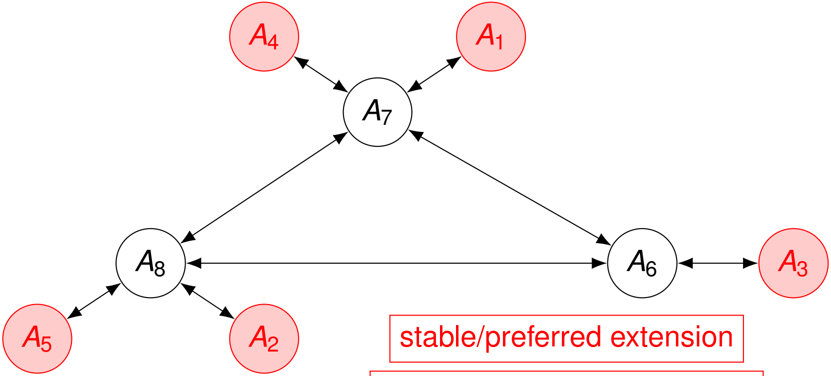

stable/preferred extension

with inconsistent conclusions

Example, Incorporation of Hypersequents

$$\mathfrak{L} = \text{HCL},\ \mathcal{R} = \{\text{Def}_{\text{H}}\},\ \mathcal{S} = \{p, q, \neg p \vee \neg q\}$$

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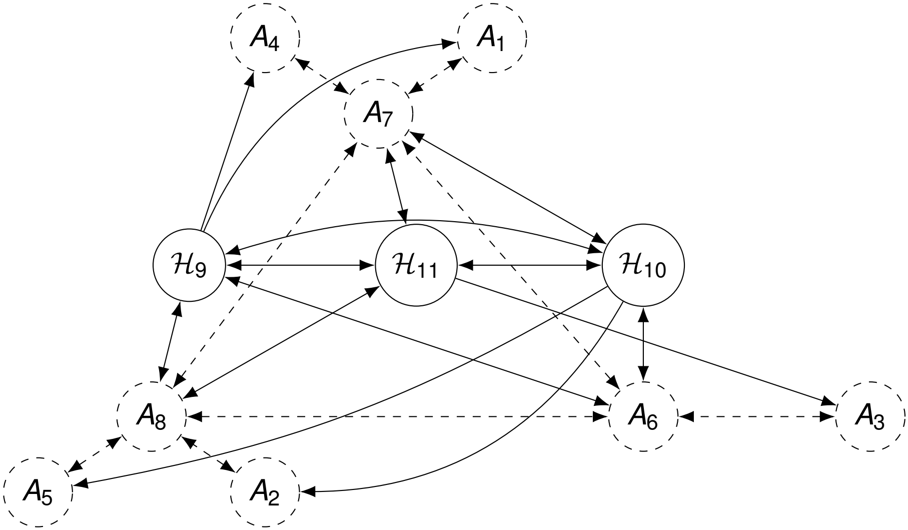
Additional hypersequents:

$\mathcal{H}_9 = \neg p \vee \neg q \Rightarrow \neg p \mid q \Rightarrow \neg p$

$\mathcal{H}_{10} = \neg p \vee \neg q \Rightarrow \neg q \mid p \Rightarrow \neg q$

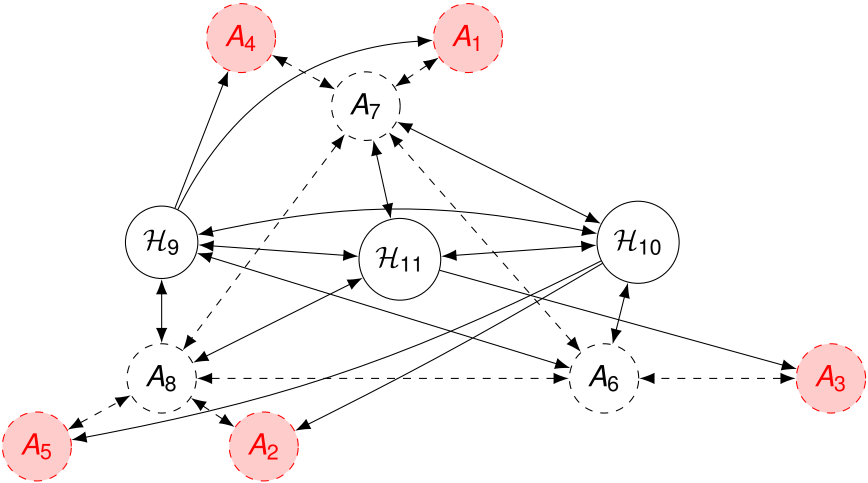
$\mathcal{H}_{11} = p \Rightarrow p \wedge q \mid q \Rightarrow p \wedge q$

Example (Cont'd.)



Dashed arrows: previous attacks, solid arrows: new attacks.

Example (Cont'd.)



Note: $\{A_1, A_2, A_3, A_4, A_5\}$ is an (inconsistent!) pref/stb extension of the sequent-based framework, but it is not admissible in the hypersequential framework (e.g., A_3 is not defended from $\mathcal{H}_{11}$). Adding an attacks of $\mathcal{H}_i$ ($i = 9, 10, 11$) make the extensions admissible (and consistent!) (e.g., $\{A_1, A_2, A_3, A_4, A_5, A_6, \mathcal{H}_{11}\}$).

(More on this in what follows)

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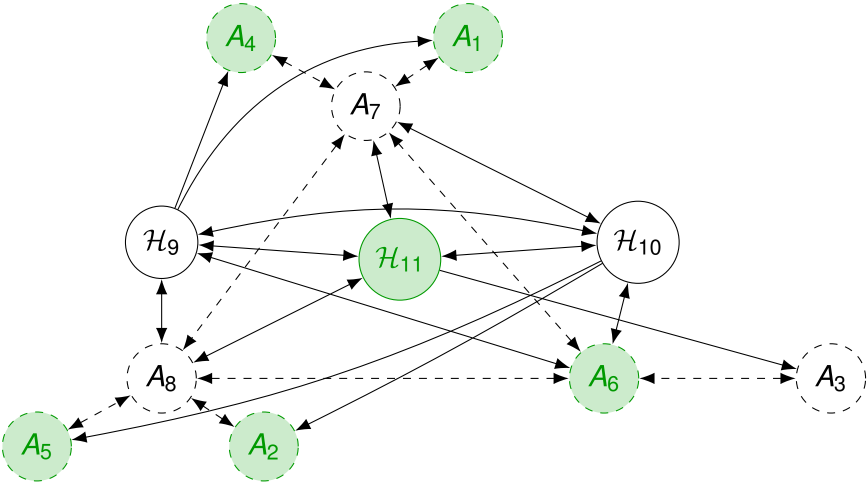

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(More on this in what follows)

Plan of Module 3

1. Knowledge Representation Considerations
   - **Extended expressive power**
     - ▷ **Distinction between strict and defeasible premises**
     - ▷ **Extending arguments with priorities**
     - ▷ **Trading sequents by hypersequents**
     - ▷ **Introducing abducitve sequents** $\Leftarrow$
   - Consistency and minimality
   - Alternative formalizations of attack rules
2. General Properties
   - Reasoning with maximal consistency
   - Rationality postulates

Abductive Reasoning with Logical Argumentation

Goal: *abductive reasoning with logic-based argumentation for providing explanations to inferences and decisions (explainable AI).*

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$\mathcal{E}$ *explains* ψ (according to $\mid\!\sim$) in the presence of $\mathcal{S}$, if:

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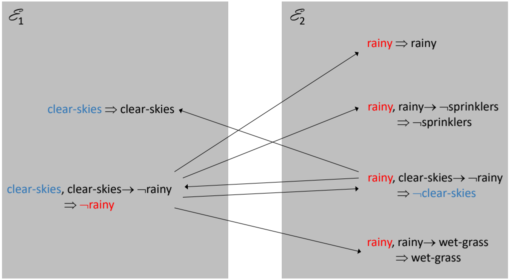
Adaptation to Argumentation: Let $\mathcal{AF}(\mathcal{S}) = \langle \text{Arg}_{\mathfrak{L}}(\mathcal{S}), \text{Attack}(\mathcal{A}) \rangle$.

$\mathcal{E}$ is a [skeptical/credulous] *external abductive explanation of* ψ, *relative to the assumptions in* $\mathcal{S}$, if:

1. $\mathcal{E}$ is $\mathfrak{L}$-consistent ($\not\vdash \neg \bigwedge_{\epsilon \in \mathcal{E}} \epsilon$),
2. ψ it does *not* [skeptically/credulously] follow from $\mathcal{AF}(\mathcal{S})$,
3. ψ [skeptically/credulously] follows from $\mathcal{AF}(\mathcal{S} \cup \mathcal{E})$.

Example

$$\mathfrak{L} = \text{CL}, \quad \mathcal{S} = \left\{ \begin{array}{l} \texttt{clear\_skies}, \quad \texttt{rainy}, \\ \texttt{clear\_skies} \rightarrow \neg\texttt{rainy}, \\ \texttt{rainy} \rightarrow \neg\texttt{sprinklers}, \\ \texttt{rainy} \rightarrow \texttt{wet\_grass}, \\ \texttt{sprinklers} \rightarrow \texttt{wet\_grass} \end{array} \right\}, \quad \text{Defeat}$$



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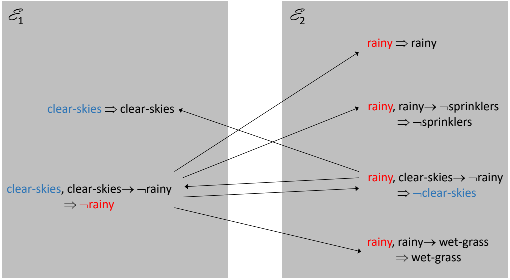
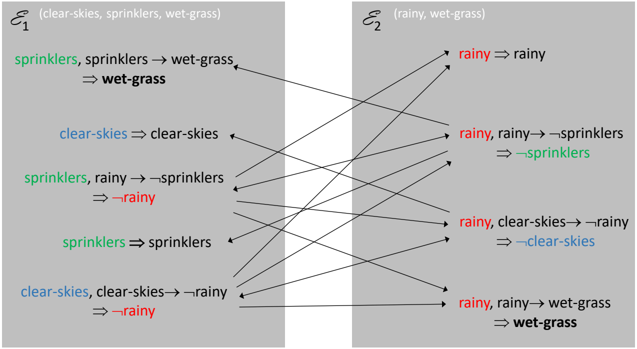


Two preferred/stable extensions. `wet_grass` is a possible (credulous) but not certain (skeptical) conclusion of $\mathcal{S}$.

How can we make `wet_grass` is a certain (skeptical) conclusion of $\mathcal{S}$?

Example (Cont'd.)

$$\mathcal{S} \cup \mathcal{E} = \left\{ \begin{array}{l} \texttt{clear\_skies},\ \ \texttt{rainy}, \\ \texttt{clear\_skies} \rightarrow \neg\texttt{rainy}, \\ \texttt{rainy} \rightarrow \neg\texttt{sprinklers}, \\ \texttt{rainy} \rightarrow \texttt{wet\_grass}, \\ \texttt{sprinklers} \rightarrow \texttt{wet\_grass} \end{array} \right\} \cup \{\texttt{sprinklers}\}$$



Example (Cont'd.)

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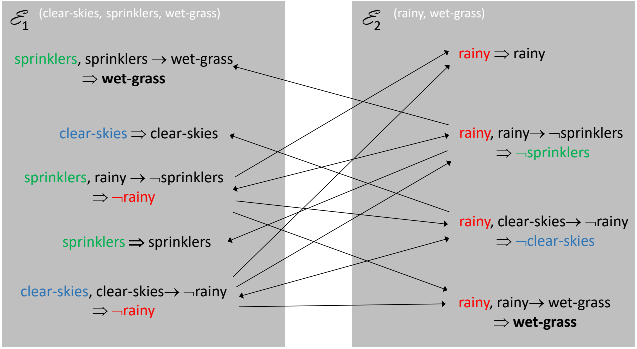

wet_grass is a certain (skeptical) conclusion of $\mathcal{S}$ with two possible (credulous) explanations: rainy and sprinklers.

Abductive Reasoning – An External View

$\mathcal{E}$ is a [skeptical/credulous] external explanation of ψ w.r.t. $\mathcal{S}$.

Consistency $\mathcal{E}$ is $\mathfrak{L}$-consistent: ($\not\vdash \neg\bigwedge_{e \in \mathcal{E}} e$)

Non-idleness ψ doesn't [skeptically/credulously] follow from $\mathcal{AF}(\mathcal{S})$,

Sufficiency ψ [skeptically/credulously] follows from $\mathcal{AF}(\mathcal{S} \cup \mathcal{E})$

Further Requirements from External Explanations

Non-vacuity $\mathcal{E} \not\vdash \psi$ (no self-explanation)

Minimality ψ doesn't [skeptically/credulously] follow from $\mathcal{AF}(\mathcal{S} \cup \mathcal{E}')$ when $\mathcal{E} \vdash \bigwedge\mathcal{E}'$ and $\mathcal{E}' \not\vdash \bigwedge\mathcal{E}$

(assuring the conciseness of the explanations)

Abductive Reasoning – An Internal View

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- The idea: In addition to 'regular sequents' (of the form $\Gamma \Rightarrow \psi$), we also introduce *abductive sequents*

$$\psi \Leftarrow [\epsilon], \Gamma$$

intuitively meaning that: *'(the explanandum) ψ may be inferred from Γ, assuming that ϵ holds'*.

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Example

$$\texttt{wet\_grass} \Leftarrow [\texttt{sprinklers}], \texttt{sprinklers} \rightarrow \texttt{wet\_grass}$$

the wetness of the grass may (also) be explained by the work of the sprinklers

Rules and Conditions for Abductive Sequents

- Abduction as 'backwards reasoning':

Abduction

$$\frac{\epsilon, \Gamma \Rightarrow \phi}{\phi \Leftarrow [\epsilon], \Gamma} \quad (\Gamma \subseteq \mathcal{S},\ \epsilon \notin \mathcal{S})$$

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- Rules for attacking abductive sequents:

Abductive Defeat

$$\frac{\Gamma_1 \Rightarrow \psi_1 \qquad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2' \qquad \psi_2 \Leftarrow [\epsilon], \Gamma_2}{\psi_2 \not\Leftarrow [\epsilon], \Gamma_2} \quad (\Gamma_2' \subseteq \Gamma_2 \cup \{\epsilon\})$$

Rules and Conditions for Abductive Sequents

- Abduction as 'backwards reasoning':

Abduction

$$\frac{\epsilon, \Gamma \Rightarrow \phi}{\phi \Leftarrow [\epsilon], \Gamma} \quad (\Gamma \subseteq \mathcal{S},\ \epsilon \notin \mathcal{S})$$

- Rules for attacking abductive sequents:

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$$\frac{\Gamma_1 \Rightarrow \psi_1 \qquad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2' \qquad \psi_2 \Leftarrow [\epsilon], \Gamma_2}{\psi_2 \not\Leftarrow [\epsilon], \Gamma_2} \quad (\Gamma_2' \subseteq \Gamma_2 \cup \{\epsilon\})$$

- In particular, we get:

Consistency

$$\frac{\rightarrow \neg\epsilon \qquad \psi \Leftarrow [\epsilon], \Gamma}{\psi \not\Leftarrow [\epsilon], \Gamma}$$

Rules and Conditions (Cont'd.)

Non-Vacuity

$$\frac{\epsilon \vdash \psi \qquad \psi \Leftarrow [\epsilon]}{\psi \not\Leftarrow [\epsilon]}$$

Non-Idleness

$$\frac{\Gamma_1 \Rightarrow \psi \qquad \psi \Leftarrow [\epsilon],\ \Gamma_2}{\psi \not\Leftarrow [\epsilon],\ \Gamma_2}$$

Minimality

$$\frac{\psi \Leftarrow [\epsilon_1], \Gamma_1 \qquad \epsilon_2 \vdash \epsilon_1 \qquad \epsilon_1 \not\vdash \epsilon_2 \qquad \psi \Leftarrow [\epsilon_2], \Gamma_2}{\psi \not\Leftarrow [\epsilon_2], \Gamma_2}$$

Abductive Sequent-based Argumentation Frameworks

An *abductive argumentation framework* for $\mathcal{S}$, a logic $\mathfrak{L}$, and a set $\mathcal{A}$ of attack rules, is a pair $\mathcal{AAF}(\mathcal{S}) = \langle \text{Arg}_{\mathfrak{L}}(\mathcal{S}), \text{Attack}(\mathcal{A}) \rangle$, where

- $\text{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of the $\mathcal{S}$-based [abductive] $\mathfrak{L}$-arguments, and
- $(A_1, A_2) \in \text{Attack}(\mathcal{A})$ iff A_1 $\mathcal{R}$-attacks A_2, for some $\mathcal{R} \in \mathcal{A}$.

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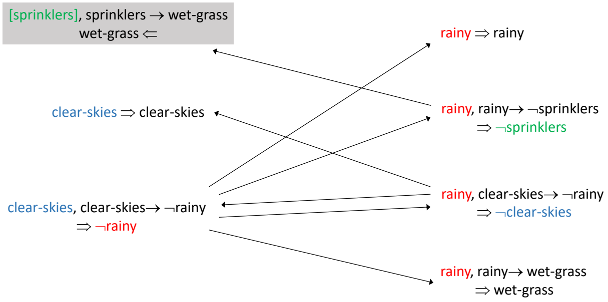

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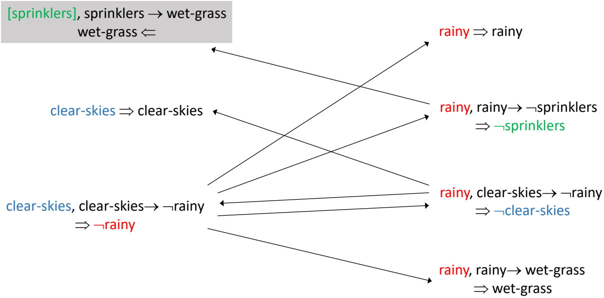
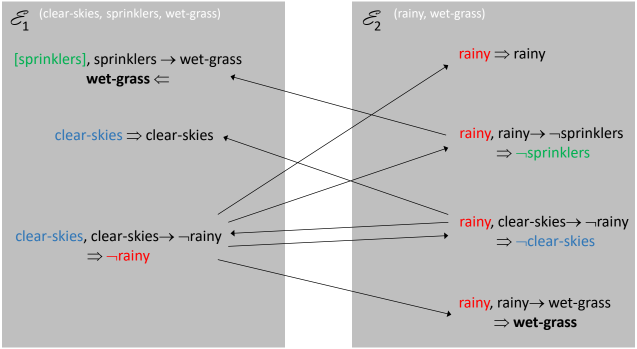


$\mathcal{E}$ is a [skeptical/credulous] *internal abductive explanation of* ψ *relative to* $\mathcal{S}$, if there is $\Gamma \subseteq \mathcal{S}$ such that $\psi \Leftarrow [\bigwedge \mathcal{E}], \Gamma$ [skeptically/credulously] follows from $\mathcal{AAF}(\mathcal{S})$.

Example (Cont'd.)

$$\mathcal{S} = \left\{ \begin{array}{l} \texttt{clear\_skies},\ \ \texttt{rainy}, \\ \texttt{clear\_skies} \rightarrow \neg\texttt{rainy}, \\ \texttt{rainy} \rightarrow \neg\texttt{sprinklers}, \\ \texttt{rainy} \rightarrow \texttt{wet\_grass}, \\ \texttt{sprinklers} \rightarrow \texttt{wet\_grass} \end{array} \right\}$$



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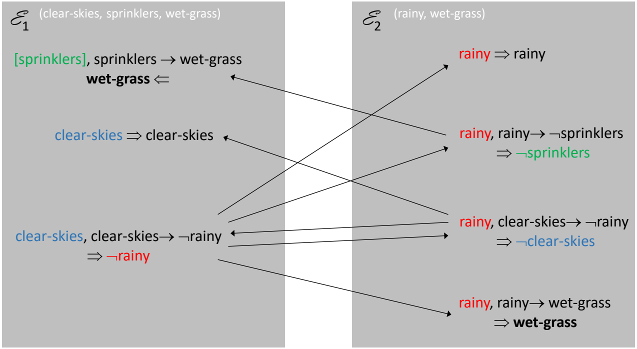

`wet_grass` is a certain (skeptical) conclusion of $\mathcal{S}$,
justified by `rainy` and the (credulous) internal explanation `sprinklers`.

Relating the Two Approaches

Let $\mathcal{E} = \{\epsilon\}$ be a singleton explanation, where Atoms$(\epsilon) \subseteq$ Atoms$(\mathcal{S})$ and let $\{\mathrm{ConUcut}\} \subset \mathcal{A} \subseteq \{\mathrm{ConUcut}, \mathrm{DirectDefeat}, \mathrm{DirectUndercut}\}$ be a set of attack rules.

Theorem

Let sem $\in \{\mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.

$\mathcal{E}$ is an external skeptical sem*-explanation of ϕ w.r.t. $\mathcal{AF}(\mathcal{S})$, based on* CL *and $\mathcal{A}$, iff $\mathcal{E}$ is an internal skeptical* sem*-explanation of ϕ w.r.t. $\mathcal{AAF}(\mathcal{S})$, based on* CL *and $\mathcal{A}^\star = \mathcal{A} \cup \{\text{Abductive Direct Defeat}\}$.*

O. Arieli, A. Borg, M. Hesse, C. Straßer. *Explainable Logic-Based Argumentation.* COMMA'22, pp.32-43, 2022.

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- The theorem holds also when Non-Vacuity is assumed.
- External and internal explanations may not coincide when Minimality is assumed.

O. Arieli, A. Borg, M. Hesse, C. Straßer. *Explainable Logic-Based Argumentation*. COMMA'22, pp.32-43, 2022.

Example

$\mathfrak{L} = \mathsf{CL}$, $\mathcal{R} = \{\mathrm{DirDef}, \mathrm{ConUcut}\}$, $\mathcal{S} = \{p, \neg p \wedge q\}$, $\mathcal{X} = \{q \wedge r \rightarrow s\}$

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$\mathfrak{L} = \mathsf{CL}$, $\mathcal{R} = \{\mathrm{DirDef}, \mathrm{ConUcut}\}$, $\mathcal{S} = \{p, \neg p \wedge q\}$, $\mathcal{X} = \{q \wedge r \to s\}$

- $q \wedge r$ is an external $\Cap$-skeptical stable explanation of s:

  If $q \wedge r$ is added to $\mathcal{S}$, the corresponding AF has two stable extensions: $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{X} \cup \{p, q \wedge r\})$ and $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{X} \cup \{\neg p \wedge q, q \wedge r\})$.

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  If, in addition, minimality is imposed, then:

- $q \wedge r$ <u>remains</u> an external $\Cap$-skeptical stable explanation of s:

  It satisfies the minimality condition.

- $q \wedge r$ <u>is not</u> an internal $\Cap$-skeptical stable explanation of s:

  One extension contains a minimality attacker of $s \Leftarrow [q \wedge r], q \wedge r \to s$, namely: $s \Leftarrow [r], \neg p \wedge q, q \wedge r \to s$.

Plan of Module 3

1. Knowledge Representation Considerations

    - Extended expressive power
        - Distinction between strict and defeasible premises
        - Extending arguments with priorities
        - Trading sequents by hypersequents
        - Introducing abducitve sequents

    (Representations of the AF ingredients:)

    - **(Forms of arguments:) Consistency and minimality**
    - (Forms of attack Rules:) Alternative formalizations of attack rules

2. General Properties

    - Reasoning with maximal consistency
    - Rationality postulates

Consistency and Minimality of Arguments' Supports

Recall from Module 2:

Definition (Besnard & Hunter)

A BH-argument based on $\mathcal{S}$ is a pair $A = \langle \Gamma, \psi \rangle$, where Γ is a minimally consistent subset of $\mathcal{S}$ such that $\Gamma \vdash_{\mathsf{CL}} \psi$.

Definition (Revised definition, for any propositional logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$)

An $\mathfrak{L}$-argument based on $\mathcal{S}$ is a $\mathcal{L}$-sequent $\Gamma \Rightarrow \psi$, where Γ is a subset of $\mathcal{S}$ and $\Gamma \vdash \psi$.

- Extensions to arbitrary (propositional) logics and languages
- The support sets need not be minimal
- The support sets need not be consistent

Let's consider in greater detail the last two issues (minimality and consistency).

Consistency and Minimality of Arguments’ Supports

Recall: We argued that support minimality and consistency may be waived from the definition of an argument. Some reasons:

- Not realistic: In real-life situations, arguments’ supports are often not minimal and sometimes even not consistent.
- Increased computational complexity: Checking inconsistency is in general (depending on the logic) an NP-complete problem, and determining minimality is a Π^2_p-complete problem for CL and at least as hard for many other logics.
- Practicalities: Minimality and consistency can be taken care of at the level of the argumentation framework themselves (by specifying appropriate attack rules and/or using suitable logics).

O.Arieli, C.Straßer, *On minimality and consistency tolerance in logical argumentation frameworks*, COMMA’20, pp.91-102, 2020.

M.D’Agostino, S.Modgil, *A fully rational account of structured argumentation under resource rounds*, IJCAI’20, pp.1841–1847, 2020.

Consistency Preservation

Arguments with inconsistent supports may not be filtered out, even in cases that the base logic is trivialized in the presence of inconsistency. Instead, inconsistency may be tolerated by attack rules of the argumentation frameworks.

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Proposition

$\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathsf{Attack}(\mathcal{A})\rangle$ *– a sequent-based AF for $\mathcal{S}$, based on a logic $\mathfrak{L}$ and a set $\mathcal{A}$ of attack rules excluding attacks on tautological arguments, that contains ConUcut.*

$\mathcal{AF}^{\mathsf{con}}(\mathcal{S}) = \langle \mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathcal{S}), \mathsf{Attack}(\mathcal{A}^{\star})\rangle$ *– a sequent-based AF for $\mathcal{S}$ where $\mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathcal{S})$ is the subset of $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ that consists only of $\vdash_{\mathfrak{L}}$-consistent arguments*†*, and $\mathcal{A}^{\star} = \mathcal{A} - \{\mathrm{ConUcut}\}$.*

$\forall \mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}, , \mathsf{SStb}\}\ \mathsf{Sem}(\mathcal{AF}(\mathsf{S})) = \mathsf{Sem}(\mathcal{AF}^{\mathsf{con}}(\mathsf{S}))$.

† More precisely, arguments that are not attacked by tautological arguments.

- Recall: ConUcut indicates that $\Gamma, \Gamma' \Rightarrow \psi_2$ is attacked when $\vdash \neg \bigwedge \Gamma'$.
- A more general version of the proposition appears in Arieli & Straßer COMMA'20 paper.

Minimization of the Arguments' Supports

Proposition

Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathsf{Attack}(\mathcal{A})\rangle$ *be a sequent-based AF for* $\mathcal{S}$ *and a* $\subseteq$*-normal set* $\mathcal{A}$ *of attack rules. We denote:*

$$\mathsf{min}_{\subseteq}(\mathcal{E}) = \{\Gamma \Rightarrow \psi \in \mathcal{E} \mid \nexists \Delta \Rightarrow \psi \in \mathcal{E} \text{ such that } \Delta \subsetneq \Gamma\}$$

$$\mathsf{Attack}^{\mathsf{min}}_{\subseteq}(\mathcal{A}) = \mathsf{Attack}(\mathcal{A}) \cap \big(\mathsf{min}_{\subseteq}(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})) \times \mathsf{min}_{\subseteq}(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}))\big),$$

$$\mathcal{AF}^{\mathsf{min}}_{\subseteq}(\mathcal{S}) = \langle \mathsf{min}_{\subseteq}(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})), \mathsf{Attack}^{\mathsf{min}}_{\subseteq}(\mathcal{A})\rangle.$$

For every $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$ *we have:*
$\mathcal{E}' \in \mathsf{Sem}(\mathcal{AF}^{\mathsf{min}}_{\subseteq}(\mathcal{S}))$ *iff* $\exists \mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$ *such that* $\mathcal{E}' = \mathsf{min}_{\subseteq}(\mathcal{E})$.

- A set of attack rules $\mathcal{A}$ is called *$\subseteq$-normal* if $(\forall \mathcal{R} \in \mathcal{A})$:

1. its attack rules are closed under $\subseteq$-stronger attacking arguments:
   if $\Gamma \Rightarrow \psi$ $\mathcal{R}$-attacks $\Delta \Rightarrow \phi$, then every $\Gamma' \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ s.t. $\Gamma' \subseteq \Gamma$, also $\mathcal{R}$-attacks $\Delta \Rightarrow \phi$.
2. its attack rules are closed under $\subseteq$-weaker attacked arguments:
   if $\Gamma \Rightarrow \psi$ $\mathcal{R}$-attacks $\Delta \Rightarrow \phi$, then $\Gamma \Rightarrow \psi$ also $\mathcal{R}$-attacks $\Delta' \Rightarrow \phi$ for every $\Delta' \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ where $\Delta \subseteq \Delta'$.

- A more general version of the proposition appears in Arieli & Straßer COMMA'20 paper (covers any preorder (i.e., any reflexive and transitive order), not only $\subseteq$).

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  Not applicable in Kleene logic KL $= \langle \{t, f, \bot\}, \{t\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$, since its does not have tautological sequents, thus there are no ConUcut-attacking arguments.

  Not applicable in Asenjo-Priest LP $= \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$, since there are no contradictory supports (every set of formulas is satisfiable), thus there are no ConUcut-attacked arguments.

Case Study: LFIs and Defeat

Defeat

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Logics of Formal Inconsistency (LFIs)

$$\text{LFI} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\circ}, \tilde{\neg}\} \rangle$$

$\circ t = t$, $\circ f = t$, $\circ \top = f$. $\circ\psi =$ 'ψ is consistent'. Thus: $p, \neg p \nvdash \circ p$.

W.Carnielli, M.Coniglio, J.Marcos, *Logics of Formal Inconsistency*, Handbook of Philosophical Logic, Vol.14, 2007.

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- When $\mathfrak{L} = \mathrm{LFI}$, $\neg p \Rightarrow \neg p$ and $p \Rightarrow p$ *should not* attack each other!
- Instead, $\neg p \Rightarrow \neg p$ may attack $p, \circ p \Rightarrow p$ and $p \Rightarrow p$ may attack $\neg p, \circ p \Rightarrow \neg p$.

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- $\Gamma_1 \Rightarrow \psi_1$ should Def-attack $\Gamma_2, \Gamma_2' \Rightarrow \psi_2$ only when $\psi_1, \neg \bigwedge \Gamma_2' \vdash \mathsf{F}$.

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Reformulation of the Defeat Rule and Its Variations

Inconsistency Defeat

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<u>Variations:</u>

Inconsistency Full Defeat

$$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1, \neg\bigwedge\Gamma_2 \Rightarrow \mathsf{F} \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}\ \text{IncFullDef}$$

Inconsistency Direct Defeat

$$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1, \neg\gamma_2 \Rightarrow \mathsf{F} \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}\ \text{IncDirDef}$$

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   $\Gamma_1 \Rightarrow \psi_1$ attacks $\Gamma_2, \Gamma_2' \Rightarrow \psi_2$ iff $\psi_1 \vdash \neg \bigwedge \Gamma_2'$ and $\Gamma_1 \vdash \circ \bigwedge \Gamma_1$.

2. **Attack based on a consistency conclusion of the attacker.**

   $\circ p, p \Rightarrow \circ p \wedge p$ attacks $\neg p \Rightarrow \neg p$, but not vice versa.
   $\circ p, p \Rightarrow p$ should *not* attack $\neg p \Rightarrow \neg p$.

   $\Gamma_1 \Rightarrow \psi_1$ attacks $\Gamma_2, \Gamma_2' \Rightarrow \psi_2$ iff $\psi_1 \vdash \neg \bigwedge \Gamma_2'$ and $\psi_1 \vdash \circ \psi_1$.

3. **Attack based on a consistency assumption of the attacked.**

   $\neg p \Rightarrow \neg p$ attacks $\circ p, p \Rightarrow p$, but not vice versa.

Defeat – Alternatives for LFIs

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3. **Attack based on a consistency assumption of the attacked.**

   $\neg p \Rightarrow \neg p$ attacks $\circ p, p \Rightarrow p$, but not vice versa.

   $\Gamma_1 \Rightarrow \psi_1$ attacks $\Gamma_2, \Gamma_2' \Rightarrow \psi_2$ iff $\psi_1 \vdash \neg \bigwedge \Gamma_2'$ and $\Gamma_2, \Gamma_2' \vdash \circ(\bigwedge \Gamma_2 \wedge \bigwedge \Gamma_2')$.

Back to Support Minimization

Remark:

According to Alternative 1 of Defeat: (previous slide)

$$\Gamma_1 \Rightarrow \psi_1 \text{ attacks } \Gamma_2, \Gamma_2' \Rightarrow \psi_2 \text{ iff } \psi_1 \vdash \neg\bigwedge\Gamma_2' \text{ and } \Gamma_1 \vdash \circ\bigwedge\Gamma_1.$$

Thus:

- $\circ p$ is not mandatory for supporting p in $\circ p, p \Rightarrow p$
- but it *is* needed for the attack of $\circ p, p \Rightarrow p$ on $\neg p \Rightarrow \neg p$
- (if a conjunction is available, $\circ p \wedge p \Rightarrow \circ p \wedge p$ has minimal support and it also attacks $\neg p \Rightarrow \neg p$, but)

  attackers may be produced more easily without the support minimization requirement.