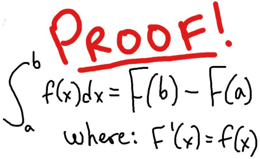
Module 4

Proof Methods



Plan of Module 4

1. **General Introduction**
   - Proof systems
   - Sequent calculi
2. Proof Systems for Logic-Based Argumentation
   - Dynamic proof systems
   - Annotation-based systems
   - Other approaches

Proof Systems

We shall use proof systems for two purposes:

1. Constructing arguments in $\text{Arg}_{\mathfrak{L}}(\mathcal{S})$ for a given base logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and a set $\mathcal{S}$ of assumptions.
2. Computing consequences of $\mid\!\sim^{\star}_{\mathfrak{L},\mathcal{A},\text{sem}}$ (for a base logic $\mathfrak{L}$, attack rules $\mathcal{A}$, semantics sem, and $\star \in \{\cap, \Cap, \cup\}$), the entailments that are induced by logic-based argumentation frameworks.

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Purpose 1 may be realized by different types of 'standard' proof systems (Hilbert-style, Gentzen-style, Tableaux methods, etc.).

Here we incorporate systems that are based on sequent calculi.[1]

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Here we incorporate systems that are based on sequent calculi.[1]

Purpose 2 is realized here by dynamic sequent calculi, allowing for non-monotonic derivation processes.

[1] Or hypersequent calculi, when arguments are represented by hypersequents.

Example: The Sequent Calculus LK for CL

Axioms: $\psi \Rightarrow \psi$

Structural Rules:

Weakening [W]: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut [C]: $$\frac{\Gamma_1, \psi \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$[\wedge\Rightarrow]$ $$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta}$$

$[\Rightarrow\wedge]$ $$\frac{\Gamma \Rightarrow \Delta, \psi \qquad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

$[\vee\Rightarrow]$ $$\frac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$$

$[\Rightarrow\vee]$ $$\frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

$[\supset\Rightarrow]$ $$\frac{\Gamma \Rightarrow \psi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$$

$[\Rightarrow\supset]$ $$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$$

$[\neg\Rightarrow]$ $$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}$$

$[\Rightarrow\neg]$ $$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

Derivations in LK

A derivation tree in LK of one side of one of De Morgan's laws:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \neg\psi, \varphi}\,[W]}{\Rightarrow \neg\psi, \neg\varphi, \varphi}\,[\Rightarrow\neg]}{\Rightarrow \neg\psi \vee \neg\varphi, \varphi}\,[\Rightarrow\vee] \qquad \dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \neg\varphi, \psi}\,[W]}{\Rightarrow \neg\psi, \neg\varphi, \psi}\,[\Rightarrow\neg]}{\Rightarrow \neg\psi \vee \neg\varphi, \psi}\,[\Rightarrow\vee]}{\Rightarrow \neg\psi \vee \neg\varphi, \psi \wedge \varphi}\,[\Rightarrow\wedge]}{\neg(\psi \wedge \varphi) \Rightarrow \neg\psi \vee \neg\varphi}\,[\neg\Rightarrow]}{\Rightarrow \neg(\psi \wedge \varphi) \supset \neg\psi \vee \neg\varphi}\,[\Rightarrow\supset]$$

- Each leaf (i.e., a most-upper line) of the tree contains an instance of the axiom schema of LK.
- The root (i.e. the bottom line) contains the proven sequent.
- Transitions from one node of the tree to another are justified by applications of the inference rules.

Another Example: Construction of Arguments in J_3^B

Recall (Module 1): $J_3^B = PAC^{F,B} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, F, B\} \rangle$

Another Example: Construction of Arguments in $\mathsf{J}_3^{\mathsf{B}}$

Recall (Module 1): $\mathsf{J}_3^{\mathsf{B}} = \mathsf{PAC}^{\mathsf{F,B}} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}, \mathsf{F}, \mathsf{B}\} \rangle$

A sound & complete sequent calculus for $\mathsf{J}_3^{\mathsf{B}}$:

- LK without the negation rules,
- Additional axiom achema: $\Rightarrow \psi, \neg\psi$
- The following logical rules for $\neg$, F, and B:

$$[\neg\neg\Rightarrow] \quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad [\Rightarrow\neg\neg] \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$[\neg\wedge\Rightarrow] \quad \frac{\Gamma, \neg\varphi \Rightarrow \Delta \qquad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\wedge] \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$$

$$[\neg\vee\Rightarrow] \quad \frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\vee] \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$$

$$[\neg\supset\Rightarrow] \quad \frac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\supset] \quad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \supset \psi), \Delta}$$

$$[\mathsf{F}\Rightarrow] \quad \Gamma, \mathsf{F} \Rightarrow \Delta \qquad [\Rightarrow\neg\mathsf{F}] \quad \Gamma \Rightarrow \Delta, \neg\mathsf{F}$$

$$[\Rightarrow\mathsf{B}] \quad \Gamma \Rightarrow \Delta, \mathsf{B}$$

$$[\Rightarrow\neg\mathsf{B}] \quad \Gamma \Rightarrow \Delta, \neg\mathsf{B}$$

- For a sound & complete calculus for $\mathsf{LP}^{\mathsf{F,B}}$, remove the rules for $\supset$.

Extension to 4All

$$4\text{All} = \langle\{t, f, \top, \bot\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\oplus}, \tilde{\otimes}, \tilde{\supset}, \tilde{\neg}, \mathsf{F}, \mathsf{B}, \mathsf{N}\}\rangle$$

Extension to 4All

$$\text{4All} = \langle \{t, f, \top, \bot\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\oplus}, \tilde{\otimes}, \tilde{\supset}, \tilde{\neg}, \mathsf{F}, \mathsf{B}, \mathsf{N}\} \rangle$$

A sound & complete sequent calculus for 4All:

- The sequent calculus for J_3^B *without* the axioms: $\Rightarrow \psi, \neg\psi$
- The following logical rules for $-$, $\otimes$, $\oplus$, and N:

$$[-\Rightarrow] \quad \frac{\Gamma \Rightarrow \Delta, \neg\psi}{\Gamma, -\psi \Rightarrow \Delta} \qquad [\Rightarrow -] \quad \frac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\psi}$$

$$[\neg - \Rightarrow] \quad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg - \psi \Rightarrow \Delta} \qquad [\Rightarrow \neg -] \quad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg - \psi}$$

$$[\otimes \Rightarrow] \quad \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \otimes \varphi \Rightarrow \Delta} \qquad [\Rightarrow \otimes] \quad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \otimes \varphi}$$

$$[\neg\otimes \Rightarrow] \quad \frac{\Gamma, \neg\psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \otimes \varphi) \Rightarrow \Delta} \qquad [\Rightarrow \neg\otimes] \quad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \otimes \varphi)}$$

$$[\oplus \Rightarrow] \quad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \oplus \varphi \Rightarrow \Delta} \qquad [\Rightarrow \oplus] \quad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \oplus \varphi}$$

$$[\neg\oplus \Rightarrow] \quad \frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \oplus \varphi) \Rightarrow \Delta} \qquad [\Rightarrow \neg\oplus] \quad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \oplus \varphi)}$$

$$[\mathsf{N} \Rightarrow] \quad \Gamma, \mathsf{N} \Rightarrow \Delta$$

$$[\neg\mathsf{N} \Rightarrow] \quad \Gamma, \neg\mathsf{N} \Rightarrow \Delta$$

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Dynamic Proof Systems

- Dynamic sequent calculus: sequent calculus + attack rules.

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  Derived formulas may be ‘discharged’ when the derivation progresses.

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- At any stage of the derivation a derived sequent may be:
  - accepted (i.e., it is derived from the premises and there is no known reason to discharge it),
  - eliminated (i.e., it is attacked by an accepted sequent), or
  - finally accepted (i.e., it is accepted and cannot be eliminated in any extension of the derivation).

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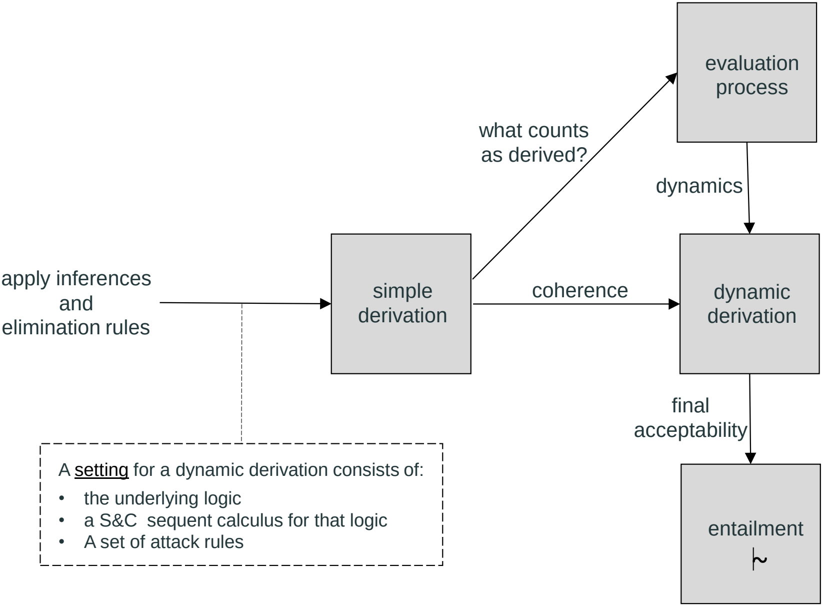
- $\mathcal{S} \vdash_{\mathsf{P}} \psi$ [$\mathcal{S} \mid\!\sim_{\mathsf{P}} \psi$] if there is a [dynamic] derivation in the [dynamic] proof system P, in which $\Gamma \Rightarrow \psi$ is finally derived for some $\Gamma \subseteq \mathcal{S}$.

Dynamic Derivations



Simple Derivations

Example: A dynamic sequent calculus based on:

- CL and its sequent calculus LK,
- $$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg\bigwedge\Gamma_2' \quad \neg\bigwedge\Gamma_2' \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2} \text{ Ucut}$$
- The assumptions $\mathcal{S} = \{p, \neg p, q\}$

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A [simple] derivation: (Sequence of tuples of the form $\langle i, \mathrm{s}, \mathrm{J}\rangle$)

1	$q \Rightarrow q$	Axiom
2	$p \Rightarrow p$	Axiom
3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$

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5	$\neg p \Rightarrow \neg p$	Axiom
!		The derived sequents in Tuples 2 and 5 are conflicting.

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4	$\Rightarrow p \vee \neg p$	$[\Rightarrow\vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$

Simple Derivations

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- CL and its sequent calculus LK,
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4	$\Rightarrow p \vee \neg p$	$[\Rightarrow\vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut 5, 5, 5, 2
!		At this point, the derived $p \Rightarrow p$ should be discharged.

A [simple] derivation (Cont'd):

1	$q \Rightarrow q$	Axiom
2	$p \Rightarrow p$	Axiom
3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
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A [simple] derivation (Cont'd):

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5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
!		... but the reverse attack may also be produced:

A [simple] derivation (Cont'd):

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3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
7	$p \Rightarrow \neg\neg p$	[...]
8	$\neg\neg p \Rightarrow p$	[...]
9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$

A [simple] derivation (Cont'd):

1	$q \Rightarrow q$	Axiom
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6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
7	$p \Rightarrow \neg\neg p$	[...]
8	$\neg\neg p \Rightarrow p$	[...]
9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$
?		What is the status of the sequents at this point?

A [simple] derivation (Cont'd):

1	$q \Rightarrow q$	Axiom
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3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
7	$p \Rightarrow \neg\neg p$	[...]
8	$\neg\neg p \Rightarrow p$	[...]
9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$

Intuition:

- $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ attack each other; they should not be finally derived.
- The sequent $\Rightarrow p \vee \neg p$ is not attacked, thus it should be finally derived.
- $q \Rightarrow q$ is defended by $\Rightarrow p \vee \neg p$, thus it should be finally derived as well.

A [simple] derivation (Cont'd):

1	$q \Rightarrow q$	Axiom
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4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
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- We need an evaluation process for determining the sequents' statuses.
- Dynamic derivation = simple derivation + evaluation process for controlling the derivation flow.

What Counts as Derived and When?
Top-Down Evaluation Algorithm

```
function Evaluate(𝒟)                          /* 𝒟 – a simple derivation */
Elim := Attack := Derived := ∅;
while (𝒟 is not empty) do {                   /* top-down iteration */
    if (Top(𝒟) = ⟨i, A, J⟩)                   /* sequent introduction tuple */
        Derived := Derived ∪ {A};
    if (Top(𝒟) = ⟨i, Ā, J⟩)                   /* sequent elimination tuple */
        if (B attacks A and B ∉ Elim)
            Elim := Elim ∪ {A};
            Attack := Attack ∪ {B};
    𝒟 := Tail(𝒟); }
Accept := Derived − Elim;
return (Elim, Attack, Accept)
```

Evaluation procedure for a simple derivation $\mathcal{D}$

The derivation (Cont'd):

	1	$q \Rightarrow q$	Axiom
	2	$p \Rightarrow p$	Axiom
	3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
	4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
	5	$\neg p \Rightarrow \neg p$	Axiom
	6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
	7	$p \Rightarrow \neg\neg p$	[...]
	8	$\neg\neg p \Rightarrow p$	[...]
$\Longrightarrow$	9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$

Evaluation:

Derived	
Elim	$\neg p \Rightarrow \neg p$
Attack	$p \Rightarrow p$
Accept	

The derivation (Cont'd):

	1	$q \Rightarrow q$	Axiom
	2	$p \Rightarrow p$	Axiom
	3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
	4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
	5	$\neg p \Rightarrow \neg p$	Axiom
	6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
$\Longrightarrow$	7	$p \Rightarrow \neg\neg p$	[...]
$\Longrightarrow$	8	$\neg\neg p \Rightarrow p$	[...]
	9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$

Evaluation:

Derived	$\neg\neg p \Rightarrow p,\ p \Rightarrow \neg\neg p,$
Elim	$\neg p \Rightarrow \neg p$
Attack	$p \Rightarrow p$
Accept	

The derivation (Cont'd):

	1	$q \Rightarrow q$	Axiom	
	2	$p \Rightarrow p$	Axiom	
	3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$	
	4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$	
	5	$\neg p \Rightarrow \neg p$	Axiom	
$\Longrightarrow$	6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$	$p \Rightarrow p$ is not eliminated since $\neg p \Rightarrow \neg p$ was eliminated
	7	$p \Rightarrow \neg\neg p$	[...]	
	8	$\neg\neg p \Rightarrow p$	[...]	
	9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$	

Evaluation:

Derived	$\neg\neg p \Rightarrow p,\ p \Rightarrow \neg\neg p,$
Elim	$\neg p \Rightarrow \neg p$
Attack	$p \Rightarrow p$
Accept	

The derivation (Cont'd):

$\Longrightarrow$	1	$q \Rightarrow q$	Axiom
$\Longrightarrow$	2	$p \Rightarrow p$	Axiom
$\Longrightarrow$	3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
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$\Longrightarrow$	5	$\neg p \Rightarrow \neg p$	Axiom
	6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$
	7	$p \Rightarrow \neg\neg p$	[...]
	8	$\neg\neg p \Rightarrow p$	[...]
	9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$

Evaluation:

Derived	$\neg\neg p \Rightarrow p,\ p \Rightarrow \neg\neg p,\ \neg p \Rightarrow \neg p,\ \Rightarrow p \vee \neg p,\ \ldots$
Elim	$\neg p \Rightarrow \neg p$
Attack	$p \Rightarrow p$
Accept	

The derivation (Cont'd):

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7	$p \Rightarrow \neg\neg p$	[...]
8	$\neg\neg p \Rightarrow p$	[...]
9	$\neg p \not\Rightarrow \neg p$	Ucut $2, 7, 8, 5$

Evaluation:

Derived	$\neg\neg p \Rightarrow p,\ p \Rightarrow \neg\neg p,\ \neg p \Rightarrow \neg p,\ \Rightarrow p \vee \neg p,\ \ldots$
Elim	$\neg p \Rightarrow \neg p$
Attack	$p \Rightarrow p$
Accept	Derived $\setminus \{\neg p \Rightarrow \neg p\}$

Dynamic Derivations

A $\langle \mathfrak{L}, \mathcal{C}, \mathcal{A} \rangle$-based *dynamic derivation* for $\mathcal{S}$ is a simple derivation $\mathcal{D}$ of one of the following forms:

- $\mathcal{D} = \langle T \rangle$, where $T = \langle 1, \mathsf{s}, \mathsf{J} \rangle$ is a proof tuple.
- $\mathcal{D}$ extends a dynamic derivation by introducing tuples whose sequents are not in Elim($\mathcal{D}$).
- $\mathcal{D}$ extends a dynamic derivation by eliminating tuples, such that:
  1. $\mathcal{D}$ is coherent [Attack($\mathcal{D}$) $\cap$ Elim($\mathcal{D}$) $= \emptyset$],
  2. ($\mathcal{S}$-based) attackers are not attacked by accepted sequents.

O.Arieli, C.Straßer: Logical argumentation by dynamic proof systems. Theoretical Computer Science 781: 63-91, 2019.

Dynamic Derivations

A $\langle \mathfrak{L}, \mathcal{C}, \mathcal{A} \rangle$-based *dynamic derivation* for $\mathcal{S}$ is a simple derivation $\mathcal{D}$ of one of the following forms:

- $\mathcal{D} = \langle T \rangle$, where $T = \langle 1, \mathsf{s}, \mathsf{J} \rangle$ is a proof tuple.
- $\mathcal{D}$ extends a dynamic derivation by introducing tuples whose sequents are not in Elim($\mathcal{D}$).
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  1. $\mathcal{D}$ is coherent [Attack($\mathcal{D}$) $\cap$ Elim($\mathcal{D}$) $= \emptyset$],
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A sequent (argument) A is *finally derived* in a dynamic derivation $\mathcal{D}$, if $T(\mathsf{s}) = \langle i, \mathsf{s}, \mathsf{J} \rangle$ is accepted in $\mathcal{D}$, and $\mathcal{D}$ cannot be extended to any derivation for $\mathcal{S}$, in which $T(\mathsf{s})$ is eliminated.

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$\mathcal{S} \mid\!\sim_{\mathsf{P}} \psi$ if there is a dynamic derivation for $\mathcal{S}$, based on $\mathsf{P} = \langle \mathfrak{L}, \mathcal{C}, \mathcal{A} \rangle$, in which $\Gamma \Rightarrow \psi$ is finally derived for some $\Gamma \subseteq \mathcal{S}$.

O.Arieli, C.Straßer: Logical argumentation by dynamic proof systems. Theoretical Computer Science 781: 63-91, 2019.

Back to the Running Example; $\mathcal{S} = \{p, \neg p, q\}$

1. $\mathcal{S} \not\mid\!\sim p$ and $\mathcal{S} \not\mid\!\sim \neg p$

$p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ attack each other so they cannot be finally accepted:

Back to the Running Example; $\mathcal{S} = \{p, \neg p, q\}$

1. $\mathcal{S} \not\approx p$ and $\mathcal{S} \not\approx \neg p$

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1	$q \Rightarrow q$	Axiom
2	$p \Rightarrow p$	Axiom
3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut $5, 5, 5, 2$

Attack : $\neg p \Rightarrow \neg p$
Elim : $p \Rightarrow p$
Accept : Derived $\setminus \{p \Rightarrow p\}$

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4	$\Rightarrow p \vee \neg p$	$[\Rightarrow\vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut 5, 5, 5, 2

Attack : $\neg p \Rightarrow \neg p$
Elim : $p \Rightarrow p$
Accept : Derived $\setminus \{p \Rightarrow p\}$

1	$q \Rightarrow q$	Axiom
2	$p \Rightarrow p$	Axiom
3	$\Rightarrow p, \neg p$	$[\Rightarrow\neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow\vee], 3$
5	$\neg p \Rightarrow \neg p$	Axiom
6	$p \not\Rightarrow p$	Ucut 5, 5, 5, 2
7	$p \Rightarrow \neg\neg p$	[...]
8	$\neg\neg p \Rightarrow p$	[...]
9	$\neg p \not\Rightarrow \neg p$	Ucut 2, 7, 8, 5

Attack : $p \Rightarrow p$
Elim : $\neg p \Rightarrow \neg p$
Accept : Derived $\setminus \{\neg p \Rightarrow \neg p\}$

Back to the Running Example; $\mathcal{S} = \{p, \neg p, q\}$

2. $\mathcal{S} \mid\!\sim p \vee \neg p$

$\Rightarrow$ $p \vee \neg p$ is derived and cannot be attacked (its support is empty), therefore it is finally accepted.

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2	$p \Rightarrow p$	Axiom
3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$

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3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
5	$p, \neg p \Rightarrow \neg q$	[...]
6	$\neg q \Rightarrow \neg q$	Axiom
7	$q \not\Rightarrow q$	Ucut $5, 6, 6, 1$

Back to the Running Example; $\mathcal{S} = \{p, \neg p, q\}$

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3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$
5	$p, \neg p \Rightarrow \neg q$	[...]
6	$\neg q \Rightarrow \neg q$	Axiom
7	$q \not\Rightarrow q$	Ucut $5, 6, 6, 1$
8	$p, \neg p \not\Rightarrow \neg q$	Ucut $4, \cdot, \cdot, 5$

Back to the Running Example; $\mathcal{S} = \{p, \neg p, q\}$

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$\Rightarrow p \vee \neg p$ is derived and cannot be attacked (its support is empty), therefore it is finally accepted.

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$q \Rightarrow \neg q$ is defended against all its $\mathcal{S}$-based attackers by a finally accepted argument, therefore it is also finally accepted.

1	$q \Rightarrow q$	Axiom	9	$p, \neg p, q \Rightarrow \neg q$	[...]
2	$p \Rightarrow p$	Axiom	10	$q \not\Rightarrow q$	Ucut 9, 6, 6, 1
3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$			
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$			
5	$p, \neg p \Rightarrow \neg q$	[...]			
6	$\neg q \Rightarrow \neg q$	Axiom			
7	$q \not\Rightarrow q$	Ucut 5, 6, 6, 1			
8	$p, \neg p \not\Rightarrow \neg q$	Ucut 4, $\cdot, \cdot$, 5			

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$q \Rightarrow \neg q$ is defended against all its $\mathcal{S}$-based attackers by a finally accepted argument, therefore it is also finally accepted.

1	$q \Rightarrow q$	Axiom	9	$p, \neg p, q \Rightarrow \neg q$	[...]
2	$p \Rightarrow p$	Axiom	10	$q \not\Rightarrow q$	Ucut $9, 6, 6, 1$
3	$\Rightarrow p, \neg p$	$[\Rightarrow \neg], 2$	11	$p, \neg p, q \not\Rightarrow \neg q$	Ucut $4, \cdot, \cdot, 9$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee], 3$			
5	$p, \neg p \Rightarrow \neg q$	[...]			
6	$\neg q \Rightarrow \neg q$	Axiom			
7	$q \not\Rightarrow q$	Ucut $5, 6, 6, 1$			
8	$p, \neg p \not\Rightarrow \neg q$	Ucut $4, \cdot, \cdot, 5$			

Back to the Running Example; $\mathcal{S} = \{p, \neg p, q\}$

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3	$\Rightarrow p, \neg p$	$[\Rightarrow\neg], 2$
4	$\Rightarrow p \vee \neg p$	$[\Rightarrow\vee], 3$
5	$p, \neg p \Rightarrow \neg q$	[...]
6	$\neg q \Rightarrow \neg q$	Axiom
7	$q \not\Rightarrow q$	Ucut $5, 6, 6, 1$
8	$p, \neg p \not\Rightarrow \neg q$	Ucut $4, \cdot, \cdot, 5$
9	$p, \neg p, q \Rightarrow \neg q$	[...]
10	$q \not\Rightarrow q$	Ucut $9, 6, 6, 1$
11	$p, \neg p, q \not\Rightarrow \neg q$	Ucut $4, \cdot, \cdot, 9$

attackers of $q \Rightarrow q$ are eliminated

Another Example

Logic with weak negation ($p \not\vdash \neg p$, $\neg p \not\vdash p$), where $p \not\vdash \neg\neg p$; Attack by Defeat.

$$\mathcal{S} = \{p, \neg p, \neg\neg p, \neg\neg\neg p, \neg\neg\neg\neg p\}, \quad A_i = \neg^i p \Rightarrow \neg^i p$$

Another Example

Logic with weak negation ($p \nvdash \neg p$, $\neg p \nvdash p$), where $p \nvdash \neg\neg p$; Attack by Defeat.

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1	A_0	Axiom
2	A_1	Axiom
3	A_2	Axiom

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1	A_0	Axiom
2	A_1	Axiom
3	A_2	Axiom
4	$\overline{A_1}$	Def, 3, 2

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3	A_2	Axiom
4	$\overline{A_1}$	Def, 3, 2
5	A_3	Axiom

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4	$\overline{A_1}$	Def, 3, 2
5	A_3	Axiom
6	$\overline{A_0}$	Def, 2, 1
7	$\overline{A_2}$	Def, 5, 3

Another Example

Logic with weak negation ($p \nvdash \neg p$, $\neg p \nvdash p$), where $p \nvdash \neg\neg p$; Attack by Defeat.

$$\mathcal{S} = \{p, \neg p, \neg\neg p, \neg\neg\neg p, \neg\neg\neg\neg p\}, \quad A_i = \neg^i p \Rightarrow \neg^i p$$

1	A_0	Axiom	
2	A_1	Axiom	
3	A_2	Axiom	
4	$\overline{A_1}$	Def, 3, 2	An Unsuccessful Attack by A_2
5	A_3	Axiom	
6	$\overline{A_0}$	Def, 2, 1	
7	$\overline{A_2}$	Def, 5, 3	

Another Example

Logic with weak negation ($p \nvdash \neg p$, $\neg p \nvdash p$), where $p \nvdash \neg\neg p$; Attack by Defeat.

$$\mathcal{S} = \{p, \neg p, \neg\neg p, \neg\neg\neg p, \neg\neg\neg\neg p\}, \quad A_i = \neg^i p \Rightarrow \neg^i p$$

1	A_0	Axiom	
2	A_1	Axiom	
3	A_2	Axiom	
4	$\overline{A_1}$	Def, 3, 2	An Unsuccessful Attack by A_2
5	A_3	Axiom	
6	$\overline{A_0}$	Def, 2, 1	
7	$\overline{A_2}$	Def, 5, 3	
8	A_4	Axiom	

Another Example

Logic with weak negation ($p \not\vdash \neg p$, $\neg p \not\vdash p$), where $p \not\vdash \neg\neg p$; Attack by Defeat.

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8	A_4	Axiom
9	$\overline{A_1}$	Def, 3, 2
10	$\overline{A_3}$	Def, 8, 5

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Logic with weak negation ($p \nvdash \neg p$, $\neg p \nvdash p$), where $p \nvdash \neg\neg p$; Attack by Defeat.

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2	A_1	Axiom	
3	A_2	Axiom	
4	$\overline{A_1}$	Def, 3, 2	
5	A_3	Axiom	
6	$\overline{A_0}$	Def, 2, 1	An Unsuccessful Attack by A_1
7	$\overline{A_2}$	Def, 5, 3	An Unsuccessful Attack by A_3
8	A_4	Axiom	
9	$\overline{A_1}$	Def, 3, 2	
10	$\overline{A_3}$	Def, 8, 5	

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9	$\overline{A_1}$	Def, 3, 2	
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8	A_4	Axiom	
9	$\overline{A_1}$	Def, 3, 2	
10	$\overline{A_3}$	Def, 8, 5	

Another Example

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Another Example

Logic with weak negation ($p \nvdash \neg p$, $\neg p \nvdash p$), where $p \nvdash \neg\neg p$; Attack by Defeat.

$$\mathcal{S} = \{p, \neg p, \neg\neg p, \neg\neg\neg p, \neg\neg\neg\neg p\}, \quad A_i = \neg^i p \Rightarrow \neg^i p$$

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8	A_4	Axiom	
9	$\overline{A_1}$	Def, 3, 2	
10	$\overline{A_3}$	Def, 8, 5	

$\mathcal{S} \mid\!\sim \neg^0 p$, $\mathcal{S} \mid\!\sim \neg^2 p$, $\mathcal{S} \mid\!\sim \neg^4 p$ while $\mathcal{S} \not\mid\!\sim \neg^1 p$ and $\mathcal{S} \not\mid\!\sim \neg^3 p$.

Derivation-Based Argumentation Frameworks

$\mathcal{AF}(\mathcal{D}) = \langle \text{Derived}(\mathcal{D}), \text{Attack}(\mathcal{D}) \rangle$: An AF induced by a [simple] derivation $\mathcal{D}$.

- $A \in \text{Derived}(\mathcal{D})$ if $\langle i, \underline{A}, \text{J} \rangle$ in $\mathcal{D}$, and
- $(A, B) \in \text{Attack}(\mathcal{D})$ if $\langle i, \overline{B}, \text{J} \rangle$ in $\mathcal{D}$ and the attacker in J is A.

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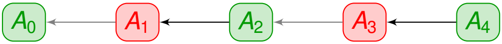
Example (from the previous slide):

1	A_0	Axiom
2	A_1	Axiom
3	A_2	Axiom
4	$\overline{A_1}$	Def, 3, 2
5	A_3	Axiom
6	$\overline{A_0}$	Def, 2, 1
7	$\overline{A_2}$	Def, 5, 3
8	A_4	Axiom
9	$\overline{A_1}$	Def, 3, 2
10	$\overline{A_3}$	Def, 8, 5



Derivation-Based Argumentation Frameworks

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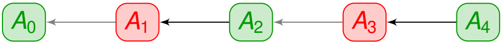

Proposition

For every simple derivation $\mathcal{D}$ the set $\text{Accept}(\mathcal{D})$ is conflict-free in $\mathcal{AF}(\mathcal{D})$.
If $\mathcal{D}$ is coherent, $\text{Accept}(\mathcal{D})$ is a stable extension of $\mathcal{AF}(\mathcal{D})$.

Some Comments and Properties of $\mid\!\sim_{\mathsf{P}}$

$\mathsf{P} = \langle \mathfrak{L}, \mathcal{C}, \mathcal{A} \rangle$ – a setting for a dynamic proof system.

(Recall: $\mathcal{C}$ is sound and complete for $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$: $\Gamma \vdash \psi$ iff $\vdash_{\mathcal{C}} \Gamma \Rightarrow \psi$)

Some properties of $\mid\!\sim_{\mathsf{P}}$ will be shown for SAC (Support Attacking, and Contrapositive) settings:

A SAC setting $\mathsf{P} = \langle \mathfrak{L}, \mathcal{C}, \mathcal{A} \rangle$ meats the following conditions:

1. $\mathcal{C}$ admits *contraposition*:

   If $\vdash_{\mathcal{C}} \Delta \Rightarrow \neg \bigwedge \Theta$, then for every $\Theta' \subseteq \Theta$ and $\Delta' \subseteq \Delta$ it holds that $\vdash_{\mathcal{C}} (\Delta \setminus \Delta') \cup \Theta' \Rightarrow \neg \bigwedge ((\Theta \setminus \Theta') \cup \Delta')$.
2. $\mathcal{A}$ consists only of attack rules in the supports of the arguments

   (Ucut, Def, ConUcut, but not Reb or their direct versions).

(Note: any calculus with $[\neg \Rightarrow]$, $[\Rightarrow \neg]$, $[\wedge \Rightarrow]$ admits contraposition)

1. Notes on Final Acceptability

Proposition

- *If A is finally accepted in $\mathcal{D}$, then it is finally accepted in any extension of $\mathcal{D}$.*

Indeed, if A is not finally accepted in an extension $\mathcal{D}'$ of $\mathcal{D}$, there is some extension $\mathcal{D}''$ of $\mathcal{D}'$ in which A is eliminated. Since $\mathcal{D}''$ is also an extension of $\mathcal{D}$, A cannot be finally accepted in $\mathcal{D}$. $\square$

Proposition

- *Let P is a SAC. If A is finally accepted in a dynamic P-derivation $\mathcal{D}$, then every dynamic P-derivation $\mathcal{D}'$ (for the same assumptions $\mathcal{S}$) can be extended to a derivation $\mathcal{D}''$ in which A is finally accepted.*

Properties of $\mid\!\sim_P$

2. $\mid\!\sim_P$ and $\vdash$

Properties of $\mid\!\sim_{\mathsf{P}}$

2. $\mid\!\sim_{\mathsf{P}}$ and $\vdash$

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If $\mathcal{A} = \emptyset$ *then* $\mid\!\sim_{\mathsf{P}} = \vdash$.

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If $\mathcal{S}$ is conflict-free (w.r.t. P), then $\mathcal{S} \mid\!\sim_{\mathsf{P}} \psi$ iff $\mathcal{S} \vdash \psi$.

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Proposition

If $\mathcal{A}$ consists only of attacks in premises (as in SACs), then:

1. $\mathrel{|\!\sim}_{\mathsf{P}} \psi$ iff $\vdash \psi$, and

2. $\mathcal{C}$ is weakly complete for $\mathrel{|\!\sim}_{\mathsf{P}}$: It holds that $\mathrel{|\!\sim}_{\mathsf{P}} \psi$ iff $\mathrel{|\!\sim}_{\mathcal{C}} \Rightarrow \psi$.

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Proposition

If P *is a SAC, then* $\mathcal{S} \mid\!\sim_{\mathsf{P}} \psi$ *iff* $\{\phi \mid \mathcal{S} \mid\!\sim_{\mathsf{P}} \phi\} \vdash \psi$.

3. $\mid\!\sim_P$ and Inconsistency Handling

1

2

3

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Proposition (no conflicts are finally derivable)

It is not the case that $\mathcal{S} \mid\!\sim_{\mathsf{P}} \psi$ *and* $\mathcal{S} \mid\!\sim_{\mathsf{P}} \neg\psi$.

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Proposition (pre-paraconsistency)

If $\vdash$ *is pre-paraconsistent* ($p, \neg p \nvdash q$) *then so is* $\mid\!\sim_{\mathsf{P}}$*.*

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Proposition (pre-paraconsistency)

If $\vdash$ *is pre-paraconsistent* ($p, \neg p \not\vdash q$) *then so is* $\mid\!\sim_{\mathsf{P}}$.

Proposition (non-interference)

If P *is a SAC and* $\mathfrak{L}$ *is uniform*[1] *then* $\mid\!\sim_{\mathsf{P}}$ *satisfies non-interference*[2] *with respect to finite sets of assumptions.*

[1] If $\mathcal{S}_1 \cup \{\phi\} \parallel \mathcal{S}_2$ and $\mathcal{S}_2$ is $\vdash$-consistent, then $\mathcal{S}_1 \vdash \phi$ iff $\mathcal{S}_1, \mathcal{S}_2 \vdash \phi$.

[2] If $\mathcal{S}_1 \cup \{\phi\} \parallel \mathcal{S}_2$, then $\mathcal{S}_1 \mid\!\sim_{\mathsf{P}} \phi$ iff $\mathcal{S}_1, \mathcal{S}_2 \mid\!\sim_{\mathsf{P}} \phi$.

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Proposition (crash resistance)

If P *is a SAC and* $\mathfrak{L}$ *is uniform, there is no* $\mid\!\sim_{\mathsf{P}}$*-contaminating set.*[3]

---

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[3] That is: there is no $\mathcal{S}$ s.t. for every $\mathcal{S}'$ where $\mathcal{S} \parallel \mathcal{S}'$ it holds that $\mathcal{S} \mid\!\sim \psi$ iff $\mathcal{S}, \mathcal{S}' \mid\!\sim \psi$.

4. NMR-Related Properties of $\mathrel{|\!\sim}_{\mathsf{P}}$

Proposition (Cumulativity [Kraus, Lehmann, Magidor, AIJ 44(1-2), 1990]**)**

$\mathrel{|\!\sim}_{\mathsf{P}}$ *is cumulative, i.e., it satisfies the following postulates:*

Cautious Reflexivity: If $\psi \not\vdash \neg\psi$ then $\psi \mathrel{|\!\sim} \psi$.

Cautious Monotonicity: If $\mathcal{S} \mathrel{|\!\sim} \phi$ and $\mathcal{S} \mathrel{|\!\sim} \psi$, then $\mathcal{S}, \phi \mathrel{|\!\sim} \psi$.

Cautious Cut: If $\mathcal{S} \mathrel{|\!\sim} \phi$ and $\mathcal{S}, \phi \mathrel{|\!\sim} \psi$, then $\mathcal{S} \mathrel{|\!\sim} \psi$.

Left Logical Equivalence: If $\phi \vdash \psi$ and $\psi \vdash \phi$, then $\mathcal{S}, \phi \mathrel{|\!\sim} \rho$ iff $\mathcal{S}, \psi \mathrel{|\!\sim} \rho$.

Right Weakening: If $\phi \vdash \psi$ and $\mathcal{S} \mathrel{|\!\sim} \phi$, then $\mathcal{S} \mathrel{|\!\sim} \psi$.

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No preferentiality. Or is violated: $\mathcal{S}, \phi \mid\!\sim \rho$ and $\mathcal{S}, \psi \mid\!\sim \rho \;\not\Rightarrow\; \mathcal{S}, \phi \vee \psi \mid\!\sim \rho$.

Counter-example (even for SAC): Consider $\mathsf{P} = \langle \mathsf{CL}, \mathit{LK}, \mathsf{Ucut} \rangle$ and $\mathcal{S} = \{p \wedge \neg q,\ p \wedge \neg r\}$. Then $\mathcal{S}, q \mid\!\sim_{\mathsf{P}} p$ and $\mathcal{S}, r \mid\!\sim_{\mathsf{P}} p$, but $\mathcal{S}, q \vee r \not\mid\!\sim_{\mathsf{P}} p$.

[See explanations in: Arieli & Straßer: Logical argumentation by dynamic proof systems, TCS 781 (2019)]

Soundness and Completeness

Theorem

Let $\mathsf{P} = \langle \mathsf{CL}, LK, Ucut \rangle$*. For a finite set* $\mathcal{S}$ *of formulas, the following are equivalent:*

- $\mathcal{S} \mid\!\sim_{\mathsf{P}} \psi$
- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{CL},\mathsf{mcs}} \psi$
- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{CL},\{\mathsf{Ucut}\},\mathsf{grd}} \psi$
- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{CL},\{\mathsf{Ucut}\},\mathsf{prf}} \psi$
- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{CL},\{\mathsf{Ucut}\},\mathsf{stb}} \psi$

Recall:

- $\mathcal{S} \mid\!\sim_{\mathsf{P}} \psi$ if $\exists \Gamma \subseteq \mathcal{S}$ s.t. $\Gamma \Rightarrow \psi$ is finally accepted in a P-derivation.
- $\mathcal{S} \mid\!\sim^{\cap}_{\mathfrak{L},\mathsf{mcs}} \psi$ if $\bigcap \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}) \vdash \psi$.
- $\mathcal{S} \mid\!\sim^{\cap}_{\mathfrak{L},\mathcal{A},\mathsf{sem}} \psi$ if $\exists A \in \bigcap \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ where $\mathsf{Conc}(A) = \psi$.

Enhancement: Sparse Final Acceptability

What about weakly skeptical semantics?

- $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathfrak{L},\mathsf{mcs}} \psi$ if $\forall \mathcal{T} \in \bigcap \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})$ it holds that $\mathcal{T} \vdash \psi$.
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Final acceptance need to be weakened as well:

$\Gamma \Rightarrow \psi$ is *sparsely finally accepted* in a derivation $\mathcal{D}$, if is it accepted in $\mathcal{D}$ and $\Gamma' \Rightarrow \psi$ (for some $\Gamma' \subseteq \mathcal{S}$) is accepted in any extension of $\mathcal{D}$.

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$\mathcal{S} \mid\!\mid\!\sim_{\mathsf{P}} \psi$, if $\Gamma \Rightarrow \psi$ ($\Gamma \subseteq \mathcal{S}$) is *sparsely* finally accepted in a P-derivation.

Why Sparse Final Acceptability?

$$\mathrm{P} = \langle \mathrm{CL}, LK, \mathrm{Ucut} \rangle, \quad \mathcal{S} = \{p \wedge q, \neg p \wedge q\}$$

1	$p \wedge q \Rightarrow p \wedge q$	*Axiom*
2	$p \wedge q \Rightarrow \neg(\neg p \wedge q)$	LK
3	$p \wedge q \Rightarrow q$	LK
4	$\neg p \wedge q \Rightarrow \neg p \wedge q$	*Axiom*
5	$\neg p \wedge q \Rightarrow \neg(p \wedge q)$	LK
6	$\neg p \wedge q \Rightarrow q$	LK
7	$\Rightarrow \neg(p \wedge q) \leftrightarrow \neg(p \wedge q)$	LK
8	$p \wedge q \not\Rightarrow q$	*Ucut*; 5, 7, 3
9	$\Rightarrow \neg(\neg p \wedge q) \leftrightarrow \neg(\neg p \wedge q)$	LK
10	$\neg p \wedge q \not\Rightarrow q$	*Ucut*; 2, 9, 6

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Neither [3] $p \wedge q \Rightarrow q$ nor [6] $\neg p \wedge q \Rightarrow q$ is finally derived, since they are respectively attacked by [5] $\neg p \wedge q \Rightarrow \neg(p \wedge q)$ and [2] $p \wedge q \Rightarrow \neg(\neg p \wedge q)$. Yet, these attacks cannot be applied *simultaneously*, since the attackers counter-attack each other. Thus $\mathcal{S} \mid\!\not\sim_{\mathsf{P}} q$.

Soundness and Completeness II

Theorem

Let $\mathsf{P} = \langle \mathsf{CL}, LK, Ucut \rangle$. *For a finite set* $\mathcal{S}$ *of formulas, the following are equivalent:*

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- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{CL},\mathsf{mcs}} \psi$
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- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{CL},\{\mathsf{Ucut}\},\mathsf{prf}} \psi$
- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{CL},\{\mathsf{Ucut}\},\mathsf{stb}} \psi$

Recall:

- $\mathcal{S} \mid\!\mid\!\sim_{\mathsf{P}} \psi$ if $\exists \Gamma \subseteq \mathcal{S}$ s.t $\Gamma \Rightarrow \psi$ is sparsely finally accepted in a P-derivation.
- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathfrak{L},\mathsf{mcs}} \psi$ if $\psi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})} \mathsf{TC}_{\mathfrak{L}}(\mathcal{T})$.
- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathfrak{L},\mathcal{A},\mathsf{sem}} \psi$ if $\forall \mathcal{E} \in \bigcap \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))\ \exists A \in \mathcal{E}$ s.t. $\mathsf{Conc}(A) = \psi$.

Plan of Module 4

1. General Introduction
   - Proof systems
   - Sequent calculi
2. Proof Systems for Logic-Based Argumentation
   - Dynamic proof systems
   - **Annotation-based systems**
   - Other approaches

Annotated Systems

Enhancements of dynamic proof systems that allow to:

1. Express in the sequent-based language the updated statuses of the arguments.
2. Express rules for status revision and for final acceptability.
3. Keep the basic properties of the dynamic proof calculi.

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1. Express in the sequent-based language the updated statuses of the arguments.
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3. Keep the basic properties of the dynamic proof calculi.

The Idea: Extending the sequents with annotations.

Annotated sequents: $\Gamma \Rightarrow^{[\mathrm{a}]} \Delta$ (or: $A[\mathrm{a}]$), where $\mathrm{a} \in \{i, e, !, *\}$

(Denoting that the sequent is introduced, eliminated, finally accepted, or a don't care condition).

Annotated Dynamic Calculi

An *annotated dynamic calculus* $\mathfrak{C}$, based on a setting $\mathsf{P} = \langle \mathfrak{L}, \mathcal{C}, \mathcal{A} \rangle$, contains the following rules:

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- Axioms and inference rules of $\mathcal{C}$, where the conditions are annotated by $[*]$ and the conclusion is annotated by [i].

$$\frac{\Gamma \Rightarrow^{[*]} \Delta, \psi \qquad \Gamma \Rightarrow^{[*]} \Delta, \varphi}{\Gamma \Rightarrow^{[\mathrm{i}]} \Delta, \psi \wedge \varphi}$$

(annotated version of $[\Rightarrow\wedge]$)

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$$\frac{\Gamma \Rightarrow^{[*]} \Delta, \psi \qquad \Gamma \Rightarrow^{[*]} \Delta, \varphi}{\Gamma \Rightarrow^{[\mathsf{i}]} \Delta, \psi \wedge \varphi}$$

(annotated version of $[\Rightarrow\wedge]$)

- Attack rules based on $\mathcal{A}$, for changing the annotations of attacked sequents from [i] to [e].

$$\frac{\Gamma_1 \Rightarrow^{[\mathsf{i}]} \psi_1 \qquad \psi_1 \Rightarrow^{[*]} \neg \bigwedge \Gamma_2 \qquad \Gamma_2, \Gamma_2' \Rightarrow^{[\mathsf{i}]} \psi_2}{\Gamma_2, \Gamma_2' \Rightarrow^{[\mathsf{e}]} \psi_2}$$

(annotated version of Defeat)

Annotated Dynamic Calculi (Cont'd.)

- Annotation revision rules:

Reactivation rules: changing annotations from [e] back to [i].

$$\frac{\Gamma_1 \Rightarrow^{[e]} \psi_1 \qquad \psi_1 \Rightarrow^{[*]} \neg \bigwedge \Gamma_2 \qquad \Gamma_2, \Gamma_2' \Rightarrow^{[e]} \psi_2}{\Gamma_2, \Gamma_2' \Rightarrow^{[i]} \psi_2}$$

(reintroducing attacked sequents whose attackers are eliminated)

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(reintroducing attacked sequents whose attackers are eliminated)

Retrospective attack rules for allowing eliminated attackers, provided that the attackers can be reactivated (handling cycles of attacks).

$$\frac{\Gamma_1, \Gamma_1' \Rightarrow^{[e]} \psi_1 \quad \psi_1 \Rightarrow^{[*]} \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow^{[i]} \psi_2}{\Gamma_2, \Gamma_2' \Rightarrow^{[e]} \psi_2}$$

(attack rule with eliminated attacker)

$$\vdots$$

$$\frac{\Gamma_3 \Rightarrow^{[e]} \psi_3 \quad \psi_3 \Rightarrow^{[*]} \neg \bigwedge \Gamma_1 \quad \Gamma_1, \Gamma_1' \Rightarrow^{[e]} \psi_1}{\Gamma_1, \Gamma_1' \Rightarrow^{[i]} \psi_1}$$

(the eliminated attacker is reactivated)

Annotated Dynamic Calculi (Cont'd.)

- Final acceptability rules: (for premise-attack rules)

Let $\mathsf{Att}(\Gamma) = \{\Delta \subseteq \mathcal{S} \mid \Delta \vdash \neg\bigwedge\Gamma\}$. Then:

$$[\mathsf{FA}_1]\ \frac{\begin{array}{l}\Gamma \Rightarrow^{[i]} \psi \\ (\forall\Delta \in \mathsf{Att}(\Gamma))\ \Delta \Rightarrow^{[*]} \neg\bigwedge\Gamma \\ (\forall\Delta \in \mathsf{Att}(\Gamma)\ \exists\Sigma \in \mathsf{Att}(\Delta))\ \Sigma \Rightarrow^{[!]} \neg\bigwedge\Delta\end{array}}{\Gamma \Rightarrow^{[!]} \psi}$$

Intuition: $\Gamma \Rightarrow \psi$ is finally accepted if: (1) it is introduced, (2) all its $\mathcal{S}$-based attackers are produced in the derivation, and (3) each such attacker is counter-attacked by a finally accepted sequent.

Annotated Dynamic Calculi (Cont'd.)

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$$[\mathsf{FA}_2]\ \frac{\Rightarrow^{[\mathsf{i}]} \psi}{\Rightarrow^{[!]} \psi} \qquad [\mathsf{FA}_3]\ \frac{\Gamma \Rightarrow^{[\mathsf{i}]} \psi \quad \mathcal{S} \notin \mathsf{Att}(\Gamma)}{\Gamma \Rightarrow^{[!]} \psi}$$

Intuition: Introduced sequents that cannot be attacked are finally accepted.

Annotated Dynamic Calculi (Cont'd.)

- Final acceptability rules: (for premise-attack rules)

Let $\mathsf{Att}(\Gamma) = \{\Delta \subseteq \mathcal{S} \mid \Delta \vdash \neg \bigwedge \Gamma\}$. Then:

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Intuition: Introduced sequents that cannot be attacked are finally accepted.

$$[\mathsf{FA}_4]\ \frac{\Gamma \Rightarrow^{[i]} \psi \quad \Gamma, \Gamma' \Rightarrow^{[!]} \phi}{\Gamma \Rightarrow^{[!]} \psi} \qquad [\mathsf{FA}_5]\ \frac{\Gamma_1 \Rightarrow^{[i]} \psi \quad \Gamma_2 \Rightarrow^{[!]} \phi \quad \Gamma_2 \Rightarrow^{[i]} \bigwedge \Gamma_1}{\Gamma_1 \Rightarrow^{[!]} \psi}$$

Intuition: If a sequent is finally derived, so is any sequent with a weaker support.

Annotated Dynamic Derivations

Derivations in an annotated dynamic calculus are a sequence of application of introduction, [retrospective] attack, and final acceptability rules, where each attack rule is followed by an annotation revision process, in which reactivation and reattack rules are applied if necessary.[4]

[4] A formal description of the revision process is given in: O. Arieli, K. van Berkel, C. Straßer: *Annotated sequent calculi for paraconsistent reasoning and their relations to logical argumentation*. Proc. IJCAI'22, pp.2532–2538, 2022.

Annotated Dynamic Derivations

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Handling Odd and Even Cycles

Recall: A derivation is $\mathcal{D}$ *coherent*, if Attack($\mathcal{D}$) $\cap$ Elim($\mathcal{D}$) $= \emptyset$.
In the annotated case this means that the end of the revision process, following an attack of an introduced sequent, the attacker is not eliminated.

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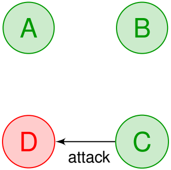

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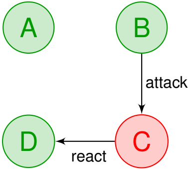

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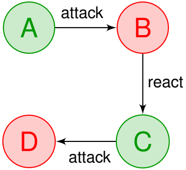

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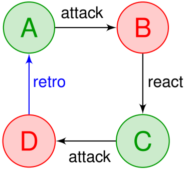

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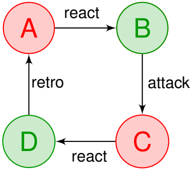

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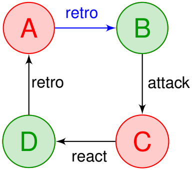

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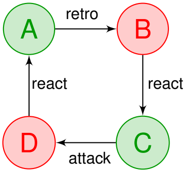

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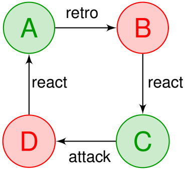

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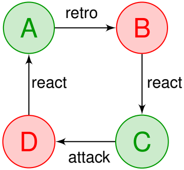

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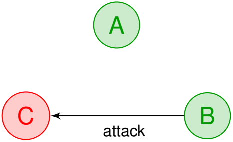

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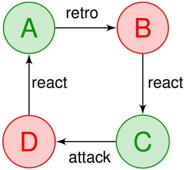

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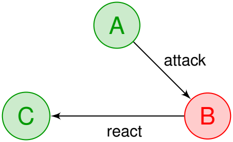

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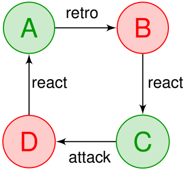

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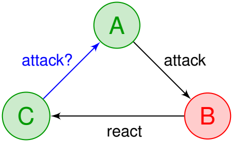

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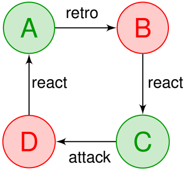

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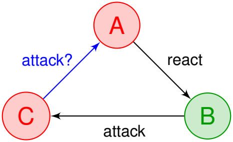

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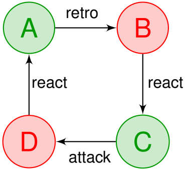

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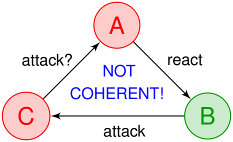

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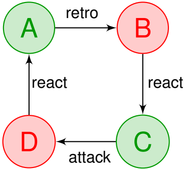

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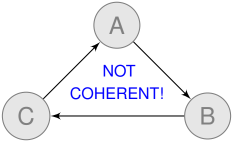

No coherent derivation is allowed.

Dynamic Derivations and the Induced AF, Revisited

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For an annotated dynamic derivation $\mathcal{D}$, let Accept($\mathcal{D}$) *be the derived sequents in $\mathcal{D}$ whose most updated status is* [i] *or* [!]. *Then:*

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Proposition

– *An annotated derivation $\mathcal{D}$ is* saturated, *if the final acceptability rules are applied to every derived sequent in $\mathcal{D}$ to which it can be applied.*

– *Let* Final($\mathcal{D}$) *be the derived sequents in $\mathcal{D}$ whose status is* [!].

If $\mathcal{D}$ is saturated, then Final($\mathcal{D}$) *is the grounded extension of $\mathcal{AF}(\mathcal{D})$.*

Plan of Module 4

1. General Introduction
   - Proof systems
   - Sequent calculi
2. Proof Systems for Logic-Based Argumentation
   - Dynamic proof systems
   - Annotation-based systems
   - **Other approaches**

References to Other Argumentative Proof Systems

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